Def. \( g: F \to G \) a morphism of fibered cats over \( C \) is an equivalence if \( \exists h: G \to F \), and base preserving natural isomorphisms \( h \circ g = \text{id}_F \) and \( g \circ h = \text{id}_G \).

Prop. \( g: F \to G \) is an equiv. (1/2) if and only if \( \forall U \in C \), \( g(U): F(U) \to G(U) \) is an equiv. of sets.

Proof. prev result \( \Rightarrow \) a fully faithful.

Conclude w/ \( h \) for each \( y \in G \), \( (Pry) = U \)

know that \( F(U) \to G(U) \) is an equiv \( \Leftrightarrow \) ess. surj.

\( \Rightarrow \) can choose \( x \in F(U) \) s.t. \( g(x) = y \)

define \( h(y) = x \), standard argument (equiv. of cats = fl + ess. surj).

Ex. If \( G \) is a prestack on \( C \), can define a fibered cat \( F \)

as follows:

- A set is a category s.t. all morphisms are identities.
- Objects of \( F \) are pairs \( (U, x) \), \( x \in G(U) \).
- Morphisms \( \text{Hom}_F((U, x), (V, y)) = \{ \text{setHom}_G(U, V) \} \).

\[ \text{Hom}(U, V) \]
Note: the cats $F(U) = \mathcal{F}(U)$ (a set!)

Rem: this gives a functor from the cat of presheaves $\mathcal{C}$
to the $(2)$-cat of fibrand categories $\mathcal{C}$

Moreover:

Prop 3.2.8: We have an equiv. of cats

$$\text{Presheaves}/\mathcal{C} \leftrightarrow \text{Categories fibr'd in Sets}/\mathcal{C}$$

Exist: this subset of fibr. cats $\mathcal{C}$ has no
not triv. 2- mono.

2- Yoneda Lemmas

Yoneda says: If $C$ a cat, \( \hat{C} \) presheaves on $\mathcal{C}$
we have functions $\mathcal{C} \to \hat{\mathcal{C}}$

$X \mapsto h_X = (y \mapsto \text{Hom}_\mathcal{C}(y, X))$

Magic $= \left( \text{Hom}_\hat{\mathcal{C}}(h_X, \mathcal{S}) \cong \mathcal{S}(X) \right)$

2-Yoneda: \[ \mathcal{C} \to \hat{\mathcal{C}} = \text{Cats fibred in } \text{Sets}/E \]

\[ \text{HOM}_\mathcal{C}(\mathcal{C}/X, F) \cong F(X) \]

Lemma: \[ \mathcal{C}/X = \text{fibred cat. associated to the presheaf } h_X. \]

Proof: $\text{Ob}(F_{h_X}) = (u, x) = (u, \alpha: u \to X) \quad F_{h_X}$

$\alpha \in h_X(u) \quad \text{Ob}(\mathcal{C}/X)$

\[ \mathcal{C}/X \to \mathcal{C} \]

$(u \to x) \mapsto u \quad \text{exercise!}$

Proof of 2.2: \[ F(X) \to \text{HOM}_\mathcal{C}(\mathcal{C}/X, F) \]

$\text{Ob}(\mathcal{C}/X) \to F(X)$

$\big{(}f: u \to x \big{)} \mapsto f^*_x \circ F(u)$

\[ \text{choose one!} \]

$(u \to x) \to \mathcal{S}(u)$
on morphisms, \( p_x(a: u \to v) \)

Why \( \mathcal{X} \) is good.

Want to describe stacks as "generalized sheaves" \( \mathcal{X} \)

associate to scheme \( u \to \mathcal{X}(u) \) = families over \( u \)

\( \text{Hom}(u, \mathcal{X}) \) category

\( v \to u \) can pullback (non canonically) \( \mathcal{X}(u) \to \mathcal{X}(v) \)

On the other hand, schemes also should correspond to stacks

\( \text{Hom stacks} \left( \text{Stack } u, \mathcal{X} \right) = \mathcal{X}(u) \)

\( \text{Fib}_{\mathcal{X}} \)

\( \mathcal{X} \)

Want: \( \text{HOM}_{\text{Sch}}((\mathcal{C}/u), \mathcal{X}) = \mathcal{X}(u) \) is 2-Yoneda!

Skip 3.3 - splittings of fibred cats
Groupoids

Def: A groupoid is a category s.t. all morphisms are isomorphisms.

Def: A fibrant cat $F \to C$ is a cat. fibred in groupoids if $\forall u \in C$, $F(u)$ is a groupoid.

"Recall" If $G, F \to C$ are fibred in sets then $\text{HOM}_C(F,G)$ is a set.

Prop: If $G, F \to C$ are fibred in groupoids, then $\text{HOM}_C(F,G)$ is a groupoid.

Proof: if $f, g \in \text{HOM}_C(F,G)$ and $\alpha : f \to g$ is a morphism, then $\alpha$ is the data of $\forall x \in F$ (base preserving natural trans)

$$
\alpha(x) : f(x) \to g(x) \in G(p(x))
$$

set $\beta : g \to f$ s.t. $\beta(x) = \alpha(x)^{-1} : g(x) \to f(x)$

This is an inverse to $\alpha$. \qed