What is a stack?

0-stack (= sheaf)
stuff on open sets
= on int's of 2 opens

1-stack (= stack)
stuff on open sets
identifications on int's of 2 opens
id's = on int's of 3 opens

2-stack
stuff on open sets
id's on int's of 2 opens
id's of id's on 3 opens
id's of id's of id's on 4 opens

∞-stacks

Basic example: assoc. to each open a top space.

\[ \text{Ex: } X \text{ top space: } Sh(X) \text{ is a stack.} \]
(means: to give a sheaf on } X, \text{ "same as" going sheaves } \mathcal{F}_{U_i}
\text{ on parts } \{U_i \to X\}, \text{ and id's } \phi_{ij}: \mathcal{F}_{U_i \cap U_j} \to \mathcal{F}_{U_i} \]
(means: to give a sm.

on an open cover \( \{ U_i \to X \} \), and id's \( q_{ij} : U_i \cap U_j \to U_i \),

s.t. \( q_{jk} q_{ij} = q_{ik} \) on \( U_i \cap U_j \cap U_k \)

\[ \text{Ex} \quad X \text{ a scheme in Zariski topo. } \mathcal{S}ch/(\mathcal{Z}/\mathcal{X}) \]

if \( U \subset X \) in \( \mathcal{O} \) open consider \( U \)-schemes.

\[ U \to U\text{-schemes} \]

\[ \text{U-scheme given by } \{ U_i \to U \} \]

\[ \{ q_i : U_i \to U \} \text{ can each } q_i \]

\[ q_{ij} : U_i \cap U_j \to U \帽{q_i} U \cap U \hat{q_j} \]

\[ + \quad q_{ij} q_{ij} = q_{ik} \]

To make sense of this, want to describe a stack \( \mathcal{X} \)

\[ \text{Open } \mathcal{X} \to \text{ "Sections" w/ notion of isom.} \]

\[ \text{Categories} \]

\[ \text{Objects in site} \]

also need restrich maps.

\[ \text{Observation/Question: Did we ever really have restrich maps anyways?} \]

\[ \text{Prototype: } \]

\[ X \]

\[ \text{family } \quad \text{"restrict } X \text{ to } U \text{"} \]

\[ V \to U \]

\[ \text{"} X \vert_V \text{" } \approx X \times_U V = f^* X \]
ugly issue: if we pick our favorite model for the fiber product then we actually won't generally have \( g^* f^* X = (f \circ g)^* X \)

\[
\begin{array}{ccc}
W & \xrightarrow{g} & V & \xrightarrow{f} & U \\
& & \downarrow & & \\
& & w & &
\end{array}
\]

but just a canonical iso morphism.

**Fibred Categories**

**Def.** C a category. A category over \( C \) is a pair \( (F, p) \) where \( F \) is a category and \( p : F \to C \) a functor.

\[
\begin{array}{c}
\exists \quad C = \text{Set/}X \\
F = \{f : y \to u \mid u \text{ open in } X\} \\
\end{array}
\]

\[
\begin{array}{ccc}
2 & \to & y \\
\downarrow & & \downarrow f \\
V & \to & U \\
\end{array}
\]

\[
p : F \to C \\
(f : y \to u) \quad \to \quad u
\]

**Def.** If \( F \to C \) is a cat. over \( C \), then \( \gamma \) morphism \( \gamma : \mathcal{X} \to \eta \) is called cartesian if for any \( \mathcal{S} \in F \)

\[
\begin{array}{c}
\mathcal{S} \to \mathcal{Z} \\
\gamma \quad \phi
\end{array}
\]

then \( \exists! \lambda : \mathcal{S} \to \mathcal{X} \) such that \( \phi \circ \lambda = \gamma \) and \( p(\lambda) = y \)

\[
\begin{array}{c}
\mathcal{S} \\
\phi
\end{array}
\]

\[
\begin{array}{c}
\mathcal{X} \quad \gamma \\
\lambda
\end{array}
\]
In this case, we say that 

\[ S \rightarrow \text{pullback of } p \text{ along } p(y) = p(y) \]  

Notation: \( p: F \rightarrow C \) is the fibred cat, we write \( F(U) \) for \( \text{cat with objects } U \) s.t. \( p(\eta) = U \), \( \eta \in F \) s.t. \( \eta \in F(U) \) and \( \text{Hom}_F(u)(\eta, \xi) = \text{Hom}_C(\eta, \xi) \) s.t. \( p(\eta) = id \).

Definition: \( p: F \rightarrow C \) is a fibred cat. if \( \forall f: U \rightarrow V \) in \( C \) \( \exists \eta \in F(V), \exists \text{cart. arrow } q: \eta \rightarrow \xi \) s.t. \( p(q) = f \).

If \( F, G \xrightarrow{P, P_0} C \) fibred cats, then a morphism \( g: F \rightarrow G \) is a functor s.t.

1) \( P_0 \circ g = P \) \( F \rightarrow G \)
2) \( g \) takes cart. arrows to cart. arrows.

If \( g \xrightarrow{g'} G \) morph. of fibred cats, then a nat. trans. \( \alpha: g \rightarrow g' \) is a base preserving nat. trans. s.t. \( \forall \xi \in F, \alpha(\xi): g(\xi) \rightarrow g'(\xi) \) then \( P_0(\alpha(\xi)) = id \).

Objects = morphisms of fibred cats.
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\[ \text{morphs} = \text{nat. trans } \text{such that } p' \to 0 \]

\( \Rightarrow \) Fibered cats are a "2-category".

**Lemma** If \( p : F \to C \) a fibered cat, and \( \gamma : S \to \mathcal{C} \) is any morphism in \( F \), then there can factor \( \gamma \) as:

\[
\begin{array}{ccc}
S & \xrightarrow{i} & \mathcal{C} \\
\downarrow \gamma & & \downarrow \lambda \\
\gamma & & F(p(\gamma))
\end{array}
\]

where \( \gamma \) is cartesian if \( \lambda \in F(p(\gamma)) \).

**Proof:** just look at a pullback \( F \) along \( p(\gamma) \).

\[
\begin{array}{ccc}
S & \xrightarrow{i} & \mathcal{C} \\
\downarrow \gamma & & \downarrow \lambda \\
\gamma & & F(p(\gamma))
\end{array}
\]

\[
\begin{array}{ccc}
p(\gamma) \rightarrow & & p(\gamma) \\
\downarrow i & & \downarrow \lambda \\
p(\mathcal{C}) & & p(\mathcal{C})
\end{array}
\]

**Lemma** If \( g : F \to G \) a morph.

of fibered cats s.t. \( \forall u \in \mathcal{C}, g(u) : F(u) \to G(u) \) is fully faithful, then \( g \) is fully faithful.

**Proof:** idea:

\[
\begin{array}{ccc}
\text{Hom}_F(S, \mathcal{C}) & \xrightarrow{i} & \text{Hom}_G(g(S), g(\mathcal{C})) \\
\downarrow & & \downarrow \\
\text{Hom}_G(F(S), G(\mathcal{C}))
\end{array}
\]

maps \( S \to \mathcal{C} \) above \( h \) \( \Rightarrow \) maps \( S \to \lambda \) \( \in F(p(S)) \)

\[
g(S) \rightarrow g(\mathcal{C}) \\
g(S) \rightarrow g(\mathcal{C}) \cup G(p(S))
\]

\( \square \)