Lecture 23: properties of morphisms of stacks, inertia and Deligne-Mumford stacks

 Lemma/Exercise

 There is a morphism of stacks on

 $C_{\text{site}}$ s.t. $\mathcal{F}$ can be equipped with a $\{T_i \to T\}$ s.t.

 $\mathcal{F}|_{T_i} \to \mathcal{G}|_{T_i}$ an equivalence. We $i \Rightarrow f$ an equivalence.

 Goal: by groups

 $X/G$, $G$ smooth gp scheme

 $\prod X/G$ cat. fibred in groupoids \{\$X \times G \to X\$

 $\prod X/G \to \{X/G\}$ via torsors

 $\prod X/G)(T) \to \{X(T) \to X_T \to \{x \in X_T(T)\}$

 objects: $X(T) \times_{G_T} \text{morphisms} \quad X(T) \times G(T)$

 Spirit of why this is an equivalence: take locally, any torsor is $G$

 Properties of morphisms

 If $P$ is any prop. of alg spaces we can say a morphism of alg stacks has prop $P$ if it is representable by alg. spaces.

 $f$ is "all spaces represent $f$ have $P$". 

 $\Rightarrow$. $\Rightarrow$. $\Rightarrow$. $\Rightarrow$.
if all spaces regular,

\[ : \mathcal{X}, T \rightarrow \mathcal{X} \]

\[ \begin{array}{c}
\text{have} \quad \downarrow \\
\text{P} \quad \rightarrow \quad \mathcal{Y}
\end{array} \]

Given a morphism of stacks, \( f : \mathcal{X} \rightarrow \mathcal{Y} \), a chart for \( f \) is an \( \mathcal{X} \)-space \( \mathcal{Y} \) as in

if \( \mathcal{D} \) is a \( \mathcal{X} \)-prop of \( \mathcal{X} \)-spaces, the \( \mathcal{Y} \)-space \( \mathcal{D} \) is smooth at \( \mathcal{Y} \) if

then we say \( (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathcal{P}) \in \mathcal{H} \) has \( \mathcal{D} \).

\[ \text{Proof: This is independent of charts.} \]

pf: common retractions.

\[ \text{Invert: } \mathcal{X} / \mathcal{S} \text{ an algebraic stack, } \]

\[ \mathcal{D}_\mathcal{X} \text{ resp. in alg. spaces} \]

\[ (x, g) \quad \sim \quad \mathcal{D}_\mathcal{X} \]

\[ g \in \text{Aut}(x) \]

\[ \begin{array}{c}
\downarrow \\
\rightarrow \mathcal{X}
\end{array} \]

\[ \text{objects } (x, g) \quad \mathcal{D}_\mathcal{X} \quad \mathcal{D}_\mathcal{X} \text{ morphs, rounds, diagrams} \]
\[
\text{M}_g \quad \text{stack objects} / T \quad \text{\text{M}}_g(T) \quad \text{smooth morphisms to} \quad C
\]

\[
T = \text{Spec } \mathbb{C}
\]

\[
\text{Aut}(C) \quad \text{flows are genus} \quad \text{g curves} \quad \text{\text{fiber}} \quad \text{\text{``curve''}}
\]

\[
\text{Adm} \leftarrow \text{Aut}(C)
\]
My \stack C T

\[ f: X \rightarrow Y \text{ ch stacks, then} \]

\[ \Delta \times Y: \Delta \longrightarrow \Delta \times \Delta \times \Delta \]

is unramified.

why? left to the reader.

\[ \begin{array}{ccc}
\Delta x & \rightarrow & X \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array} \]

\[ \Rightarrow \text{ can talk about } f \text{ is } q \text{-compact, } q \text{-rep.} \]

\[ \text{Def: An algebraic stack is Deligne-Mumford if } \]

\[ \text{étale surjection } X \rightarrow \Delta \text{ where } X \text{ is a scheme.} \]

\[ \text{Thm: } \mathcal{X}/S \text{ algebraic stack is } \Delta M \iff \Delta \times \mathcal{X} \times \mathcal{X} \]

is formally unramified.