Definitions of algs. stacks

\[ \mathcal{X} \text{ a stack s.t. } \mathcal{X} \text{ rep. by spaces} \]

\[ \exists X \rightarrow \mathcal{X} \text{ sm. surj. x alg. space} \]

\[ \mathcal{X} = \text{stackification of } \{X_0/X_1\} \text{ rep. groupoid given by} \]

\[ X_1 \xrightarrow{t} X_0 \text{ spaces} \]

\[ s, t, \text{ smooth} \]

Example: Stack Quotients

\[ X/S \text{ algebraic space } G/S \text{ smooth \& p. scheme act. on } X \]

\[ \left[ X/G \right] \text{ "stack quotient"} \]

Definition 1:

\[ \left[ X/G \right] = \text{stackification of } \{X/G\} \]

\[ G \times X \xrightarrow{\pi_2} X \]

\[ \text{Def 2:} \]

\[ \left[ X/G \right] \text{ objects: triples } (T, P, \pi) \]

\[ \pi: P \rightarrow X_T \text{ \& equivalent } \]

\[ \text{morphisms: } (T, P, \pi) \rightarrow (T', P', \pi') \]

\[ T \rightarrow T' \]

\[ \text{s.t. comp. with } \pi' \text{'s. } \]
This is a stack since it is a lift construction.

Does it representable? i.e., \( \text{Isom}((P, \pi), (P', \pi')) \), space?

By going local to proper if 

then we can assume \( P = P' = G_T \)

\[
\begin{align*}
G_T & \to X_T \\
G(1) \to & \to \text{Isom} \\
G_T & \to G_T
\end{align*}
\]

\[
\text{space} \to \downarrow (\pi, \pi' \circ g) \\
X_T & \to X_T \times X_T
\]

Exist: \( G/S \) smooth alg. scheme \( B_G = B_{sG} = [S/G] \) (inv. action)

"Classifying space" by above \( \text{Hom}(T, B_G) \leftrightarrow \text{Category of } G_T \text{torsors.} \)

Prop if \( X, Y, Z \) alg. stacks \( S \)

8.1.16 \[ \]
If sketch: 
\[(\text{fibred cats in groupoids, then})
\]
\[(\text{y''} \rightarrow y') \rightarrow y\]
\[\text{equivalence}\]

Proof of prop:

\[\text{known}
\]
\[\text{\(\mathcal{X} \times \mathcal{Y}\) is a stack}
\]

need to show: A rep., E. g.:

given objects \(\alpha = (x, y, o)\) \(\rightarrow (x', y', o')\) in \(\mathcal{X} \times \mathcal{Y}(T)\)

want to show \(\text{Isom}(\alpha, \beta)\) is an algebraic space.
Lemma: pullback of smooth surj. is smooth surj.

Def: If $P$ is a property of $S$ schemes, stable in the smooth topology, then we say an alg. stk. has prop. $P$ if
\( \mathcal{E} \times \mathcal{E} \xrightarrow{\pi} \text{sm. surj}., X \text{ a scheme w/ prop P.} \)

(†)

Ex: loc. noeth., regular, loc. finite type, loc. finite presentation.