Stein Factorization: \(X, Y \subseteq \text{alg. spaces}\)

Let \(f : X \to Y\) 2. compact, 2. sep. morphism, we say \(f\) is Stein if \(\mathcal{O}_Y \to f^* \mathcal{O}_X\) is an iso. of sheaves.

(ex: proper morph of varieties then Stein \(\iff\) fibers are connected) (Taniuki)

Main Observation: \(f : X \to Y\) 2. comp., 2. sep. then \(f\) has a natural factorization

\[
\begin{align*}
    X & \xrightarrow{a} X' \xrightarrow{b} Y \\
    f & \quad (\text{Stein}) \quad (\text{affine})
\end{align*}
\]

\[
X' = \text{Spec}_Y (f^* \mathcal{O}_X)
\]

Ex: application:

\[
f : X \to Y\] sep., 2. finite.

Stein Factorization:

\[
\begin{align*}
    X & \xrightarrow{g} Z \xrightarrow{h} Y \\
    f & \quad (\text{open})
\end{align*}
\]
Theorem (Chow's Lemma)

$S$ noeth scheme $\xrightarrow{f} S$ sep morph. finite type
$X$ reduced alg. space. Then $\mathbb{Z} \rightarrow X$ proper birational
s.t. $X$ is a q. projective $S$-scheme.

Theorem (Finiteness of Cohomology)

If $f: X \rightarrow Y$ proper morphism of loc. noeth. alg. spaces
$\mathcal{F}$ coherent sheaf on $X$, then $R^q f_* \mathcal{F}$ are coherent
on $Y \neq g \neq 0$.

(Recall $g = \text{Spec } k$, $R^q f_* \mathcal{F}$ a $k$-vector space, called $\text{H}^q(X,F)$
coherent $\Leftrightarrow H^q(X,F)$ finite dimensional)

Proof by induction on top space $|X|$

Omitting Representability shift

e.g. "Algebraization of formal moduli", etc.
Artin
Recall: Given a groupoid in schemes $\mathcal{X}_1 \rightrightarrows \mathcal{X}_0$ one can form an associated stack $[\mathcal{X}_0/\mathcal{X}_1]$ (over the big étale site).

**Def.** If $\mathcal{X}_1 \rightrightarrows \mathcal{X}_0$ is smooth, then one can form $[\mathcal{X}_0/\mathcal{X}_1] = \text{stackification} \Delta$ of representable fibered cat in groupoids $[\mathcal{X}_0/\mathcal{X}_1]$,

$$[\mathcal{X}_0/\mathcal{X}_1](T) = \text{hom}_{\mathcal{X}_0}(T) \ar_{\text{mor}} \mathcal{X}_1(T)$$

(António)

An algebraic stack is a stack equivalent to one as above.

**Def.** A morphism $f : \mathcal{X} \to Y$ of stacks over $\text{Et}(S)$ is called representable if $U \in \text{Sch}/S$ and $g : U \to Y$ the fiber product $\mathcal{X}_U \times_Y U$ is an algebraic space.
Lemma \[ f : X \to Y \text{ as above is representable} \Rightarrow \]

\[ + \quad V \to Y, \text{ } V \text{ alg. space, } \exists x y V \text{ an alg. space.} \]

\[ \text{pt: } \leq \leq \]

\[ \Rightarrow \quad ? \quad \text{given } V \to Y, \text{ can } U \to V \text{ a scheme get } x \in U \]

\[ \exists \text{ scheme } x \]

\[ \exists x y U \to \exists x y V \to X \]

\[ \downarrow \]

\[ \uparrow \]

\[ \exists U \to V \to Y \]