Let $X_i \xrightarrow{\phi_i} X_0$ groupoid in schemes $(X_1 \to X_0 \times X_0)$

then $s: X_1 \to T$ is invariant if the composition $X_1 \xrightarrow{\phi_i} X_0 \to T$ are equal

$s: X_1 \to T$ universally invariant if it's invariant and for all $g: X_0 \to T'$ invariant, we have

$X_0 \xrightarrow{g} T' \xrightarrow{T} T$. 

Thm 6.2.2 assume that $X_i \xrightarrow{\phi_i} X_0$ groupoid w/s,t, finite flat and for any $x \in X_0$, $s(t^{-1}(x))$ is contained in an affine of $X_0$. then $T$ a universal invariant morphism $X_0 \to T$.

In fact, it will be universal for loc. ringed spaces.

Consequence (?)

Thm 6.4.1 $X/s$ alg. space, q-sep. then $\exists V$ scheme, $\phi$ a dense open embedding $V \to X$. 

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and a dense open embedding \( \nu : Y \to X \).

**Remark:** Spaces are birrtille to schemes.

Can talk about function fields (i.r.d.), etc.

In fact, given an algebraic space, associate a top space.

**Quasi-Coherent sheaves & Coherent sheaves**

Let \( X \) be a space. The small etale site of \( X \) has underlying category the set of etale morphisms of algebraic spaces \( Y \to X \) & covers = families \( \{ \nu_i : Y_i \to Y \} \) s.t. \( \nu_i \to Y \) is surjective.

\( \text{Et}(X) \) site. \( X_{\acute{e}t} \) topos.

Variations: Big etale site (all algebraic space morphisms for underlying cat).

- \( \text{Et}^{t}(X) \subset \text{Et}(X) \) subcat of morphisms \( Y \to X \) y scheme.

\( X_{\acute{e}t} = X_{\acute{e}t} \) is an equiv. of topos!

To define sheaf on \( X_{\acute{e}t} \) suffices to define one on \( X_{\acute{e}t} \)
\[ R \Rightarrow U \to X \]

\[ \varphi(x) = \mathcal{E}_0 (\varphi(U) \Rightarrow \varphi(R)) \]

\[ \Downarrow \varphi(u) \Rightarrow \downarrow \varphi(u) \]

\[ \exists i : \Theta \times_t X_{\Theta} \text{ via sheet } \Theta^i \text{ in } X_{\Theta} \]

\[ \text{via } \Theta^i(t) = \Theta_t(T) \]

\[ \text{sheet on } \mathcal{S}(\mathcal{S}/\mathcal{S})_{\Theta} \circ (\mathcal{S}(\mathcal{S}/\mathcal{S})_{\Theta}) \text{ on } \]

\[ X_{\Theta} \Rightarrow X_{\Theta} \]

How to define sheaves on \( X \)?

\[ R \overset{i}{\Rightarrow} U \to X \]

\[ \text{= sheet on } U \]

\[ \text{together w/ } \exists : s^* T \to t^* S \text{ i.e. } U \to X \]

\[ s.t. \quad \pi_{23}^* \otimes \pi_{12}^* \otimes = \pi_{13}^* \text{ effective descent } \]

\[ \text{on } U \times_t U \times_t U \]

\[ \text{why? this gives sheaves on } \mathcal{E}(\mathcal{S}) \text{ via descent}. \]

\[ \# T \to X \text{ scheme} \]

\[ \overset{i}{\Rightarrow} \]

\[ U \times_t T \to U \]

\[ \overset{i}{\Rightarrow} \]

\[ R \times_t T \to R = U \times_t U \]

\[ (U \times_t T) \times_t (U \times_t T) \overset{i}{\Rightarrow} \]

\[ R \times_t T \to R = U \times_t U \times_t U \]
Punchline: \(\text{stuf} \circ \chi = \text{stuf} \circ \text{Et}(\mathcal{X})\) 
\((\text{Et}(\mathcal{X})) = \text{desc. data} \circ \text{def} \) wrt \(U \rightarrow X\)

Def: an \(\mathcal{O}_X\) module \(M\) on \(X_{\text{et}}\) is quasi-coherent if \(E \rightarrow X\) s.t. \(M_{\mathcal{U}}\) is \(\mathcal{O}_U\) coherent over \(U\).

Prop: If \(f: X \rightarrow Y\) is a morph. of alg. spaces, get induced morphism \(f^*\) to pair \(f: X_{\text{et}} \rightarrow Y_{\text{et}}\) via: \(\text{Et}(Y) \rightarrow \text{Et}(X)\) 
\((U \rightarrow Y) \rightarrow (U \times_Y X \rightarrow X)\)

\(f^* M = f^{-1} M \otimes f^{-1}(O_Y) O_X\)

\(f^* M\) induced by \(\Theta_Y \rightarrow f_\* \Theta_X\)

Prop 7.1.19: If \(f: X \rightarrow Y\) morph. of alg. spaces \(/S\)

a) \(M\) \(\mathcal{O}_Y\) coh on \(Y\) \(\Rightarrow f^* M\) \(\mathcal{O}_X\) coh on \(X\)

b) \(f\) quasi-compact, quasi-separated, \(N\) \(\mathcal{O}_Y\) coh on \(X\) \(\Rightarrow f^* N\) \(\mathcal{O}_Y\) coh on \(Y\)

Affine Morphisms
$X$ alg. space, $A$ 2 coh. sheaf of commutative $\mathcal{O}_X$-algebras,
define $\text{Spec}_X(A)$ via

$$\text{Spec}_X(A)(T) = \{ (f, \xi) \mid f: T \to X, f^*A = \mathcal{O}_T, \xi \text{ $\mathcal{O}_T$-alg. morph} \}$$

$$\xymatrix{\text{Spec}_X(A) & \text{Spec}(A(U)) \\
X \ar[r] & U \ar[u] \\
& X \ar[u]}$$

Proof: $\text{Spec}_X(A)$ is an algebraic space & the natural morphism $\text{Spec}_X(A) \to X$ is affine.
eq. of cat. s

$$\text{algebra morphisms } U \to X$$

$$\text{eq. coh. sheaves of $\mathcal{O}_X$-algebras } A$$

Applications:

- Maximal reduced subspace
  $X$ alg. space, $N \subset \mathcal{O}_X$ subobject of locally nilp. functions,
is a set of ideals, $X_{\text{red}} = \text{Spec}_X(\mathcal{O}_X/N)$
. Scheme-theoretic closure

\( f : X \rightarrow Y \) any space

\( \text{Spec} y(\mathcal{O}_Y/K) \)

\( K = \ker(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X) \)

tens on \( Y \), which vanish on \( X \)