Math 477, Practice sheet for Exam 2
Solutions

1. Suppose n numbers $X_1, X_2, \dots, X_n$ are chosen from a uniform distribution on $[0, 10]$. We say that there is an increase at i if $X_i < X_{i+1}$. Let I be the number of increases. Find $E[I]$.

Let E_i be the event that there is an increase at i.

$$P(E_i) = \frac{1}{100}\int\int_{x_i<x_{i+1}} dx_i dx_{i+1} = \frac{1}{100}\int_0^{100}\int_0^{x_{i+1}} dx_i dx_{i+1} = \frac{1}{100}\int_0^1 00x_{i+1}dx_{i+1} = \cdots = 1/2$$

Let I_i be the indicator variable for E_i, and let $X = \sum I_i$. We want to find $E[X]$. But we have:

$$E[X] = E[\sum_{i=1}^{n-1} I_i] = \sum E[X_i] = \sum P(E_i) = (n-1)(1/2).$$

2. Suppose that the time until a hurricane in months in a particular region in a given year is represented by an exponential random variable X with density function $f(x) = 12e^{-12x}$. Suppose the time to rebuild after a hurricane is given by a random variable Y uniformly distributed on the interval $[3, 8]$. Find the expected time elapsed until a hurricane occurs and rebuilding is complete.

We want to find $E[X + Y] = E[X] + E[Y]$. We have

$$E[X] = \int_0^\infty xf(x) = 12\int_0^\infty xe^{-12x}dx = \cdots = 1/12$$

and

$$E[Y] = \int_3^8 xdx = \cdots = 11/2$$

so $E[X + Y] = 1/12 + 11/2$.

Note that this is a very sad problem, since they are getting 12 huricanes in a month and it takes 5.5 months to rebuild...

3. Suppose that X and Y are uniformly distributed independent random variables in $[0, 1]$. Find $E[X^2+Y^2]$.

$$E[X^2 + Y^2] = \int_0^1\int_0^1 (x^2 + y^2)dxdy = \int_0^1 \left[x^3/3 + xy^2\right]_0^1 dy = \int_0^1 (1/3 + y^2)dy = \cdots = 2/3$$

4. Suppose numbers A, B, C are picked independently and uniformly from the interval $[0, 1]$. What is the probability that the equation $Ax^2 + Bx + C = 0$ has two real roots?

This is $P(B^2 - 4AC > 0)$. Note that since $A, B, C \in [0, 1]$, if we want $B^2 > 4AC$ we need to also have $AC < 1/4$, and therefore $A < \frac{1}{4C}$. Of course, if $C < 1/4$ then this is no condition on A since $A < 1$ already. We can therefore describe the region given by

$$A, B, C \in [0, 1] \text{ and } B^2 - 4AC > 0$$

as

$$\{C \in [0, 1/4], A \in [0, 1], B \in [2\sqrt{AC}, 1]\} \cup \{C \in [1/4, 1], A \in [0, \frac{1}{4C}], B \in [2\sqrt{AC}, 1\}$$

and so we get

$$P(B^2 - 4AC > 0) = \int_{C=0}^{C=1/4} \int_{A=0}^{A=1} \int_{B=2\sqrt{AC}}^{B=1} dBdAdC + \int_{C=1/4}^{C=1} \int_{A=0}^{A=\frac{1}{4C}} \int_{B=2\sqrt{AC}}^{B=1} dBdAdC$$

For the first integral, we have:

$$\begin{aligned}
\int_{C=0}^{C=1/4} \int_{A=0}^{A=1} \int_{B=2\sqrt{AC}}^{B=1} dBdAdC &= \int_{C=0}^{C=1/4} \int_{A=0}^{A=1} \Big[B\Big]_{B=2\sqrt{AC}}^{B=1} dAdC \\
&= \int_{C=0}^{C=1/4} \int_{A=0}^{A=1} 1 - 2\sqrt{AC} dAdC \\
&= \int_{C=0}^{C=1/4} \Big[A - 2\sqrt{C}\frac{2}{3}A^{3/2}\Big]_{A=0}^{A=1} dC \\
&= \int_{C=0}^{C=1/4} 1 - \frac{4}{3}\sqrt{C} dC \\
&= \Big[C - \frac{4}{3}\frac{2}{3}C^{3/2}\Big]_0^{1/4} \\
&= \frac{1}{4} - \frac{8}{9}(\frac{1}{4})^{3/2} \\
&= \frac{1}{4} - \frac{8}{9}(\frac{1}{2})^3 \\
&= \frac{1}{4} - \frac{1}{9} = \frac{5}{36}
\end{aligned}$$

and for the second, we have:

$$\begin{aligned}
\int_{C=1/4}^{C=1}\int_{A=0}^{A=\frac{1}{4C}}\int_{B=2\sqrt{AC}}^{B=1} dBdAdC &= \int_{C=1/4}^{C=1}\int_{A=0}^{A=\frac{1}{4C}} \Big[B\Big]_{B=2\sqrt{AC}}^{B=1} dAdC \\
&= \int_{C=1/4}^{C=1}\int_{A=0}^{A=\frac{1}{4C}} 1-2\sqrt{AC}dAdC \\
&= \int_{C=1/4}^{C=1} \left[A - 2\sqrt{C}\frac{2}{3}A^{3/2}\right]_{A=0}^{A=\frac{1}{4C}} dC \\
&= \int_{C=1/4}^{C=1} \frac{1}{4C} - \frac{4}{3}C^{1/2}\left(\frac{1}{4C}\right)^{\frac{2}{3}} dC \\
&= \int_{C=1/4}^{C=1} \frac{1}{4C} - \frac{4}{3}C^{1/2}\left(\frac{1}{8}\right)C^{-2/3}dC \\
&= \int_{C=1/4}^{C=1} \frac{1}{4C} - \frac{1}{6}C^{-1/6}dC \\
&= \frac{1}{4}\ln C - \frac{1}{6}\left(\frac{6}{5}\right)C^{5/6}\Big]_{C=1/4}^{C=1} \\
&= [\frac{1}{4}\ln 1 - \frac{1}{5}] - [\frac{1}{4}(-\ln 4) - \frac{1}{5}(1/4)^{5/6}] \\
&= -\frac{1}{5} + (1/4)2\ln 2 + \frac{1}{5}(1/4)^{5/6}]
\end{aligned}$$

The final answer is therefore

$$5/36 - 1/5 + (1/2)\ln 2 + (1/5)(1/4)^{5/6} \sim 0.35 \sim 1/3$$

5. Suppose that a game is played with a group of $2n$ people twice. Each time, the players are randomly paired. Let X be the number of pairs which occur in both the first and second game. Find $E[X]$.

6. If you roll a fair die, what is the expected number of rolls necessary in order to get a '1'?

   Since there is a 1/6 chance each roll, the expected time is (by the geometric random variable argument) 6.

7. If you roll a fair die, what is the expected number of rolls necessary in order to get both a '1' and a '2' (not necessarily in that order)?

   There is a $2/6 = 1/3$ chance of getting either a 1 or a 2. As in the previos problem, the expected time to get one or the other is 3. After this, there is a 1/6 chance each roll of getting the remaining number. Therefore 6 more rolls are expected. The answer is then $6 + 3 = 9$.