Math 477, (a few solutions to) Final Exam Practice Sheet

1. Given 4 randomly chosen people,

   (a) what is the probability that they are all born on different days of the week?

   This is the probability that the second person was born on a different day as the first person (6/7), times the probability that the third person was born on a different day than the first or second (5/7), times the probability that the fourth was born on a different day than the first, second or third (4/7). So the answer is: (6/7)(5/7)(4/7).

   (b) Let X be the random variable representing the number of pairs of people born on the same day of the week. Find $E[X]$.

   For a given pair of people, there is a 1/7 chance that they are born on the same day of the week. Let $E_{i,j}$ be the event that the i'th and j'th person were born on the same day. We then have $P(E_{i,j}) = 1/7$, and if $I_{i,j}$ is the indicator variable for $E_{i,j}$, we have $X = \sum I_{i,j}$. Therefore

   $$E[X] = E[\sum I_{i,j}] = \sum E[I_{i,j}] = \sum P(E_{i,j}) = \binom{4}{2} 1/7 = 6/7.$$

2. Suppose a random variable X represents the result of a die roll (possible values $1, 2, 3, 4, 5, 6$), and Y is the random variable which is 0 if the result of the roll is even and 1 if it is odd.

   Compute the covariance $Cov(X, Y)$.

   We have $Cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$.
   $E[X] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 35/6 = 7/2$ and $E[Y] = \frac{1}{6}(1 + 0 + 1 + 0 + 1 + 0) = 3/6 = 1/2$.
   $E[XY] = \frac{1}{6}(1 + 0 + 3 + 0 + 5 + 0) = 9/6 = 3/2$
   So, $Cov(X, Y) = 3/2 - (7/2)(1/2) = 6/4 - 7/4 = -1/4$.

3. Suppose that a person under 10 years of age has a 15% chance of having their shoelaces untied, and a person 10 years of age or older has a 3% chance of having their shoelaces untied. If 15% of the population is under the age of 10, what is the probability that a person is under the age of 10 given that their shoelaces are untied?

4. Suppose X is a random variable with probability density function $f(x) = 2 - 2x$ for $0 \le x \le 1$.

   (a) Compute the moment generating function for X.

   (b) Compute $Var(X)$.

5. Suppose X and Y are random variables with $Var(X) = 4$, $Var(Y) = 1$ and $Var(X - Y) = 3$. Is it possible that X and Y are independent? Why or why not?

6. Suppose that we are given a die which is possibly not fair, and that we don't know the likelihood of any of the results $\{1, 2, 3, 4, 5, 6\}$. Let X be the random variable representing the result of a roll. If we know that $E[X] = 3$, show that $P(X = 6) \le 1/2$.

7. Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ be the cumulative distribution function of the unit normal random variable.

   Suppose that a coin is flipped 1000 times, and let X be the number of heads. Use the central limit theorem to estimate the probability that $X < 550$. You may give your answer in terms of Φ.

   Let Y_i be the random variable describing the result of the i'th flip – this is 1 if the result is a head, and 0 if it is a tails. We first compute $\mu = E[Y_i] = \frac{1}{2}(0+1) = 1/2$. Since $Y_i^2 = Y_i$, we have $\sigma^2 = Var(Y_i) = E[Y_i^2] - E[Y_i]^2 = 1/2 - 1/4 = 1/4$. We have the Y_i's are identical and independent, $X = \sum Y_i$, and so:

   $$P(X < 550) = P\left(\sum Y_i < 550\right) = P\left(\sum Y_i - (1/2)1000 < 50\right) = P\left(\frac{\sum Y_i - 500}{(1/2)\sqrt{1000}} < \frac{50}{(1/2)\sqrt{1000}}\right)$$

   and by the central limit theorem, this is approximately equal to $P\left(Z < \frac{50}{(1/2)\sqrt{1000}}\right)$, where Z is the unit normal random variable. This can be written as

   $$\int_{-\infty}^{\frac{50}{(1/2)\sqrt{1000}}} \Phi(x)dx$$

8. Let X be a random variable with mean μ and variance σ^2. Show that $P(|X - \mu| > 2\sigma) \leq 1/4$.

9. Suppose that the temperature (in Fahreinheit) on a given day is a random variable given by a uniform distribution on $[0, 100]$. In a sequence of 7 days, what is the expected number of days which are colder then the day before them?

   This question is badly worded. The only days which should be under consideration are the days 2,3, 4, 5, 6, 7, since only these days have days before them to compare. For the i'th day, let E_i be the event that day i is colder than day $i-1$. We have $P(E_i) = 1/2$ (since this is the same as the probability that $i-1$ is colder than i, as all are independently chosen). Let X be the number of days which are colder than the day before them. We then have $X = \sum I_i$ where I_i is the indicator variable for E_i. We get:

   $$E[X] = E[\sum I_i] = \sum E[I_i] = \sum P(E_i) = 6(1/2) = 3$$

10. Suppose that the temperature (in Fahreinheit) on a given day is a random variable given by a uniform distribution on $[0, 100]$. What is the probability that the temperature on a given day is colder than both the day before and the day after.

    Consider the temperature on these three days, and call these A, B, C. Since they are equally likely to appear in any order, the coldest of the three days is equally likely to arise in any of the three days. In particular, since only one of the three spots results in the desired outcome, there is a $1/3$ chance that the coldest day is in the middle, and so a $1/3$ chance that a given day is colder than both the day before and the day after.

11. Suppose that the temperature (in Fahreinheit) on a given day is a random variable given by a uniform distribution on $[0, 100]$. If the temperature on day 2 is colder than day 1, what is the probability that the temperature on day 2 is also colder than on day 3?

12. On a given day, the expected number of times that I forget something important is 3. Suppose that this is described by a Poisson random variable.

    (a) What is the probability that on a given day I forget 5 things?

    (b) What is the probability that on a given day I forget 5 things if we know that I have forgotten at least 4 things?

13. Suppose that on day 1, $3n$ people are divided up into n teams of 3 people each. On day 2, the people are randomly re-assigned into possibly different teams of 3, any grouping being equally likely.

Let X be the number of people who have some teammate in common on both day 1 and day 2. What is the expected value of X?

Let E_i be the event that the *ith* person has a teammate in common on both days. We can calculate $1 - P(E_i)$, the probability that they have no teammates in common, as the probability that the two randomly chosen teammates are among the $3n - 3$ people who were not in their team on the first day, out of all the $3n - 1$ other people. This is

$$1 - P(E_i) = \frac{\binom{3n-3}{2}}{\binom{3n-1}{2}} = \frac{(3n-3)(3n-4)}{(3n-1)(3n-2)}$$

and so $P(E_i) = 1 - \frac{(3n-3)(3n-4)}{(3n-1)(3n-2)}$. Let I_i be the indicator variable for E_i. Then we have $X = \sum I_i$. We have

$$E[X] = E[\sum I_i] = \sum E[I_i] = \sum P(E_i) = (3n)\left(1 - \frac{(3n-3)(3n-4)}{(3n-1)(3n-2)}\right)$$