Math 477, (a few solutions to) Exam 2

Name: ____

Net ID: ____

Please hand only only clearly written work, not scratch paper. Clearly mark your final answers for each problem and clearly explain your reasoning. Full credit will only be given on problems for which your work is clearly shown.

The only allowable materials for this exam are paper, pens and pencils. No notes, textbooks or calculators will be allowed.

Numerical answers do not need to be simplified.

This exam has 6 questions, (and a bonus question!) for a total of 30 points from the standard questions.

Honor Pledge (sign before handing in)

On my honor, I have neither received nor given any unauthorized assistance on this examination

Signature ____

1. (5 points) Suppose X and Y are uniformly distributed independent random variables with $0 \leq X \leq 1$ and $1 \leq Y \leq e$. Find $E[X/Y]$.

$$
\begin{aligned}
E[X/Y] &= \frac{1}{e-1} \int_0^1 \int_0^e x/y dy dx \\
&= \frac{1}{e-1} \int_0^1 x \left[\log y\right]_0^e dx \\
&= \frac{1}{e-1} \int_0^1 x \left[1-0\right] dx \\
&= \frac{1}{e-1} \left[\frac{1}{2}x^2\right]_0^1 \\
&= \frac{1}{2(e-1)}
\end{aligned}
$$

2. (5 points) Suppose that you have a group of n people, all of whom have different ages. Suppose that they are arranged in a random order in a circle. Let X be the number of people who are younger then both people standing next to them. What is the expected value of X?

Let E_i be the event that the person standing in the i'th spot is younger than both people next to them. To compute $P(E_i)$, Consider the person in the i'th spot as well as the people on either side of them. Call these three people A, B, C. There are $3! = 6$ ways these people can be ordered, each of which is equally likely, and in exactly two of these orderings, the youngest person is in the middle. This means that $P(E_i) = 2/6 = 1/3$.

Let I_i be the indicator variable for E_i and $X = \sum I_i$. The expected number of people younger then both the people next to them is then $E[X] = E[\sum I_i] = \sum E[I_i] = \sum P(E_i) = \sum 1/3 = n/3$.

3. (10 points) Suppose that X and Y are random variables with joint density function of the form

$$f(x,y) = \begin{cases} C(x+y) & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant C.
We have

$$\begin{aligned} 1 = \int\int f(x,y)dxdy &= C\int_0^2\int_0^2 (x+y)dxdy \\ &= = C\int_0^2 \left[\frac{1}{2}x^2 + xy\right]_0^2 dy \\ &= = C\int_0^2 (2+2y)\,dy \\ &= = C\left[2y + y^2\right]_0^2 dy \\ &= = C\,[4+4]\,dy \end{aligned}$$

so $C = 1/8$.

(b) What is the probability that $X < 1$ given that $X < Y$?
$P(X < 1|X < Y) = P(X < 1 \text{ and } X < Y)/P(X < Y)$. Now, $P(X < Y) = \int\int_{x<y} f(x,y) = \int_0^2\int_x^2 f(x,y)dydx$ and $P(X < 1 \text{ and } X < Y = \int_0^1\int_x^2 f(x,y)dydx$. We now compute these:

$$\begin{aligned} P(X < Y) &= \int_0^2\int_x^2 (x+y)dydx \\ &= \frac{1}{8}\int_0^2 \left[xy + \frac{1}{2}y^2\right]_x^2 dx \\ &= \frac{1}{8}\int_0^2 \left[(2x+2) - (x^2 + \frac{1}{2}x^2)\right] dx \\ &= \frac{1}{8}\left[x^2 + 2x - \frac{1}{2}x^3\right]_0^2 = \frac{4+4-4}{8} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(X < Y \text{ and } X < 1) &= \int_0^1\int_x^2 (x+y)dydx \\ &= \frac{1}{8}\left[x^2 + 2x - \frac{1}{2}x^3\right]_0^1 = \frac{5}{16} \end{aligned}$$

So, $P(X < 1|X < Y) = 10/16 = 5/8$.

(c) Find $E[X|Y = y]$.
We have $E[X|Y = y] = \frac{\int xf(x,y)dx}{\int f(x,y)dx}$.

$$\frac{1}{8}\int_0^2 x(x+y)dx = \frac{1}{8}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2y\right]_0^2 = \frac{1}{8}\left[\frac{1}{3}8 + \frac{1}{2}4y\right] = \frac{1}{3} + \frac{y}{4}$$

$$\frac{1}{8}\int_0^2 (x+y)dx = \frac{1}{8}\left[\frac{1}{2}x^2 + xy\right]_0^2 = \frac{1}{8}\left[2+2y\right] = \frac{1}{4} + \frac{y}{4}$$

$$E[X|Y = y] = \frac{\frac{1}{3} + \frac{y}{4}}{\frac{1}{4} + \frac{y}{4}} = \frac{4+3y}{3+3y}$$

4. (5 points) Suppose that X and Y are independent identical random variables with density function $f(x) = 3e^{-3x}$, $x > 0$. Find the density function for the variable $X + Y$.

5. (5 points) If you roll a fair die, what is the expected number of rolls necessary in order to get the same number twice?

For example, if you roll: 1, 4, 3, 5, 4, then it would have taken 5 rolls

Let E_i be the event that there is a repeated number first at roll i. Then $P(E_i)$ is the probability that the first $i-1$ rolls are different, multiplied by the probability that the i'th roll is one of the first $i-1$ results. The probability that the first 2 rolls are different is 5/6, that the first 3 rolls are different is (5/6)(4/6), etc. Assuming that the first $i-1$ rolls are different, the probablity that the i'th roll results in a prior roll is $(i-1)/6$. All together, we have

$$P(E_i) = (\frac{5}{6})(\frac{4}{6})\cdots(\frac{5-i+1}{6})(\frac{i-1}{6})$$

Let X be the number of rolls it takes. We then have

$$\begin{aligned} E[X] = \sum iP(X=1) = \sum iP(E_i) \\ = 2(1/6) + 3(5/6)(2/6) + 4(5/6)(4/6)(3/6) + 5(5/6)(4/6)(3/6)(4/6) \\ + 6(5/6)(4/6)(3/6)(2/6)(5/6) + 7(5/6)(4/6)(3/6)(2/6)(1/6)(1) \end{aligned}$$

6. (5 points (bonus)) Two points P_1, P_2 are randomly chosen in $[0,1]$, dividing up the interval into three smaller intervals.

(a) What is the expected length of the line segment containing 0?

Let X, Y be the location of P_1, P_2, and let L be the length of the segment containing 0. We have $E[L] = E[L|X < Y]P(X < Y) + E[L|Y < X]P(Y < X)$. Since $P(X < Y) = P(Y < X)$, both are $\frac{1}{2}$, and we have $E[L] = E[L|X < Y]$. But if $X < Y$, $L = X$. So we have

$$E[L] = E[X|X < Y] = \frac{\int_0^1 \int_0^y x dx dy}{\int_0^1 \int_0^y dx dy} = \frac{\int_0^1 \int_0^y x dx dy}{\int_0^1 \int_0^y dx dy} = \frac{1/6}{1/2} = 1/3$$

(b) What is the probability that the three segments can form the three sides of a triangle?