Theme: Riemann-Roch

Classical RR (curves over fields)

Hirzebruch RR (proj varieties over fields)

Groth. RR (proj. morphisms of varieties)

Panin RR — " — rel to morphisms of
(+ Smirnov) oriented cohom theories

"Recall" definitions of K-theory (K_0) (Notation of Manin)

Def: $K^\circ(X) = \dfrac{\text{free ab gp gen by iso classes of loc free finite rk sheaves on } X}{\langle [E]-[E']-[E''] \mid \text{ses } 0 \to E'' \to E \to E' \to 0 \rangle}$

(also $K(X)$)

$[E]$

Ab group but also has a ring structure:

$[E]\cdot[F] = [E \otimes F] \qquad 1 = [\mathcal{O}_X]$

$K^\circ(X) \longrightarrow \mathbb{Z}$

$[E] \longmapsto \operatorname{rk} E$

recall: $X = \operatorname{Spec} A$

E/X loc. free

$\longleftrightarrow E = \tilde{M}$

M projective A-mod

Def $K_0(X) = \dfrac{\text{free Ab gp gen. by iso classes of coherent sheaves}}{\text{same relation } \langle [E]-[E']-[E''] \mid \text{ses. } 0 \to E'' \to E \to E' \to 0 \rangle}$

E/X coherent $\longmapsto$ $E = \widetilde{M}$

M is a f.g. A-mod.

$X = \operatorname{Spec} A$

A = Noetherian

[A comm ring then M/A is projective $\overset{\text{f.g. } A \text{ Noeth.}}{\iff}$ $\forall\, p \in \operatorname{Spec} A$ $M \otimes_A A_p$ is a free A_p mod]

loc. free	coh	q. coh
$E = \widetilde{M}$	$E = \widetilde{M}$ f.g.	$E = \widetilde{M}$
M proj	(A Noeth.)	M an A-mod.

Problem w/ ring structure for $K_0(X)$

is that $\otimes$ need not respect relation!

$$0 \to E'' \to E \to E' \to 0$$

$$[E] = [E''] + [E']$$

$$[E]\cdot[F] = [E''][F] + [E'][F]$$

i.e. want $0 \to E''\otimes F \to E\otimes F \to E'\otimes F \to 0$

to be exact.

i.e. want $\underline{\mathrm{Tor}}_1^{\mathcal{O}_X}(E', F) = 0$

ok if E' or F is flat ($\Leftarrow$ f. rank, loc. free)

Actually shows: $K_0(X)$ is a $K^0(X)$ module.

Useful feature of $K_0(X)$ is that it has

"fundamental cycles" of subschemes

$Z \hookrightarrow X$ closed $\quad [Z] \in K_0(X)$

$$[\mathcal{O}_Z] = [\mathcal{O}_X/\mathcal{I}_Z]$$

If X is $\overset{\text{Noeth.}}{\text{regular}}$, turns out (regular $\Rightarrow$ Noeth) (for many)

$$K_\bullet(X) \cong K^\bullet(X) \; !$$

we always have a map $K^\bullet X \longrightarrow K_\bullet X$

$$[E] \longmapsto [E]$$

why is this surjective?

Theorem of Serre: X Noeth. scheme is regular $\Longleftrightarrow$ every coherent sheaf E/X has a finite locally free resolution

$$0 \to F_r \to F_{r-1} \to \cdots \to F_1 \to F_0 \to E \to 0$$

F_i loc. free.

can show in this case that in $K_\bullet X$

$$[E] = \sum_{i=0}^{r} (-1)^i [F_i]$$

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & & & \\
 & & K_1 \arrow[dr, hook] & & & & \\
 & F_2 \arrow[ur] \arrow[rr] & & F_1 \arrow[rr] \arrow[dr] & & F_0 \arrow[r] & E \arrow[r] & 0 \\
 & & & 0 \arrow[r] & K_0 \arrow[ur] \arrow[dr] & & \\
 & & & & & 0 &
\end{tikzcd}$$

in this case, get ring structure on $K_\bullet(X)$.

Intersections work (when you're lucky) in this way:

i.e. if $W, Z \hookrightarrow X$ which are "transverse"

i.e. $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_Z) = 0 \quad i > 0 \Rightarrow$

$$[W] \cdot [Z] = [W \cap Z]$$

sheaf theoretic $\cap$

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

$$\otimes \mathcal{O}_W = \mathcal{O}_X / \mathcal{I}_W$$

$$\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{O}_W, \mathcal{O}_Z) \to \mathcal{O}_W \otimes \mathcal{I}_Z \to \mathcal{O}_W \to \mathcal{O}_W \otimes \mathcal{O}_Z \to 0$$

$$\mathcal{O}_W \otimes \mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_W \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_Z = \mathcal{O}_X / \mathcal{I}_Z + \mathcal{I}_W$$

__Product__ in $K.X$

if we have coherent sheaves E, E'

make resolutions

$$0 \to F_r \dashrightarrow F_0 \to E \to 0$$

$$0 \to F_s' \dashrightarrow F_0' \to E' \to 0$$

$$[E] \longleftrightarrow \sum (-1)^i [F_i] \qquad [E][E'] = \sum (-1)^i [F_i \otimes E']$$

$$0 \to F_r \otimes E' \to \cdots \to F_1 \otimes E' \to F_0 \otimes E' \to 0$$

not exact, just a complex.

similar hom. alg. spliting shows

$$\sum (-1)^i [F_i \otimes E'] = \sum (-1)^i \operatorname{Tor}_i^{\mathcal{O}_X}(E, E')$$

in particular, if $\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_Z) = 0 \quad i > 0$

$$\Rightarrow [\mathcal{O}_W][\mathcal{O}_Z] = [\mathcal{O}_W \otimes \mathcal{O}_Z] = [\mathcal{O}_{W \cap Z}]$$

Relate rgs $K(X) = K_\cdot(X) = K^\cdot(X)$

X smooth variety / field k

and $CH(X)$

Ab. gp maps
$$CH(X) \longrightarrow K(X)$$
$$[Z] \longrightarrow [\mathcal{O}_Z]$$

(well defined on rat'l equivalence)

Consider divisors $D \subset X$ Cartier

what's the relationship between

$[\mathcal{O}_D] \in K(X)$

an $[\mathcal{O}(D)]^\vee$

$$\mathcal{I}_D = \mathcal{O}(-D) = \mathcal{O}(D)^\vee$$

$$0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

$\mathcal{I}_D = \mathcal{O}(D)^\vee$

$$[\mathcal{O}_D] = 1 - [\mathcal{O}(D)^\vee]$$

"First chern class map in K-thy"

$L \rightsquigarrow 1 - [L^\vee] = [\mathcal{O}_D]$ K-thy

invertible sheaf

$L \longrightarrow c_1(L) = [D]$ Chow

Our next goal: build a ring map backwards

$$K(X) \longrightarrow CH(X)$$ using chern classes.

we noticed before that we can define

"total chern classes"

$$E/X \rightsquigarrow 1 + c_1(E) + c_2(E) + \dots + c_r(E) = c(E)$$

vb. rank E

these had the property that

$$0 \to E'' \to E \to E' \to 0$$

$$\Rightarrow \quad c(E) = c(E'')c(E')$$

$$[E''] + [E'] \xrightarrow{c} c(E'')c(E')$$

"standard" yoga to turn this into a ring map using powerseries (exponential)

Next time: we tweak c to get

$$ch : K(X) \longrightarrow CH(X) \otimes \mathbb{Q}$$

ring hom.

to head towards Riem. Roch.

RR: computes global sections of line bundles on curves.

L/X $\quad$ X curve.

$\Gamma(L)$ in terms of $\deg L$ & geometry of X

$$\dim H^0(L) - \dim H^1(L) = \deg L + 1 - g$$

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ \operatorname{Spec} k \end{tikzcd} \qquad H^0(L) = \pi_* L \qquad (H^1(L) = R^1\pi_* L)$$

$$c_1(L) \in CH^1(X) = CH_0(X)$$

$$\deg L = \pi_* c_1(L)$$

$$\begin{tikzcd} L \arrow[d, bend right] & K(X) \arrow[r, "c_1"] \arrow[d] & CH(X) \arrow[d] & c_1 L \arrow[d] \\ \Gamma(L) & K(pt) \arrow[r] & CH(pt) & \deg L \end{tikzcd}$$

$K(pt) = \mathbb{Z}$, $CH(pt) = \mathbb{Z}$