First topic for today: Normal cones

Check out Fulton's appendix B

$A_*(X) = CH_*(X)$

Motivational discussion:

Given $V, W \hookrightarrow X$ closed subvarieties

want to define $[V]\cdot[W] \in CH_*(X)$

in nice circumstances want $[V]\cdot[W] = [V \cap W]$

e.g. transverse intersection.

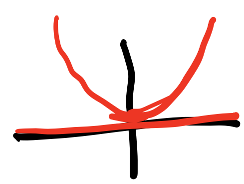
Not too bad even if intersection isn't transverse, if we consider $V \cap W$ scheme-theoretically.

e.g. $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$ $\quad V = Z(x) \quad W = Z(y)$

$V \cap W = Z(x,y)$ = pt at origin

$[V]\cdot[W] = [V \cap W] = [pt]$

$V = Z(y - x^2) \quad W = Z(y)$



set theoretic $\cap$

$[pt]$

$y = 0$

$y - x^2 = 0$

$x^2 = 0$

$x = 0$

$V \cap W$ = subscheme cut out by the sum of ideal sheaves

$\mathcal{I}_V + \mathcal{I}_W$

Recall: If R local ring consider length of R as module over itself ⟹ Fulton's length

$= Z(y - x^2, y) = Z(y, x^2)$

length 2 scheme ⟹ Fulton says

$R = \left(\frac{k[x,y]}{y, x^2}\right)_{(x,y)}$ $[Z(y,x^2)] = 2[Z(x,y)]$

Victory? No.

What about if $\cap$ has wrong dimension?!

$W = Z(x)$ $V = Z(x)$

Transversal = undefined for now

but motivational

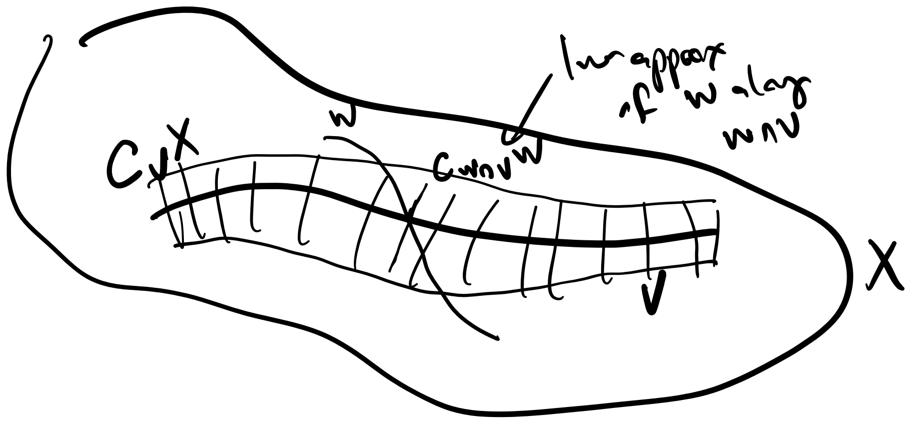
$R/I \otimes R/J$ as expected

something to do with tor?

Normal cone strategy:

1. replace intersection $W \cap V \subset X$ to one of form $(C_{W\cap V} W) \cap (\text{zero section})$

$\nearrow$ linearization of $W \cap C_V X$



Def Given $V \subset X$ define

$$C_V X = \mathrm{Spec}_{\mathcal{O}_X} \bigoplus_n \mathcal{I}_V^n / \mathcal{I}_V^{n+1}$$

$\shortparallel$

Normal cone of V in X

Compare various related objects

$$Bl_V X = \mathrm{Proj}_{\mathcal{O}_X} \bigoplus_n \mathcal{I}_V^n$$

Exceptional divisor of blowup $Bl_V X|_V$

$$= \mathrm{Proj}_{\mathcal{O}_X} \bigoplus_n \mathcal{I}_V^n / \mathcal{I}_V^{n+1}$$

fact: If $\mathscr{I}_V$ is loc. cut out by reg. seq. sequence, then the $\mathcal{O}_X$ algebra $\bigoplus \mathscr{I}_V^n/\mathscr{I}_V^{n+1} \cong \mathrm{Sym}^{\bullet}_{\mathcal{O}_X}(\mathscr{I}_V/\mathscr{I}_V^2)$

(actually an $\mathcal{O}_V$-alg)

i.e. locally on V this looks like $\mathbb{A}^N_V$ $N = \text{rank } \mathscr{I}_V/\mathscr{I}_V^2$ on V.

$= \text{codim of } V \text{ in } X$

in this special case, we write

$$N_V X \equiv C_V X$$

$\mathscr{I}_V/\mathscr{I}_V^2$

= conormal bundle (Hartshorne)

$$C_{W \cap V} W \subset C_V X$$

$$\underset{V}{\mathbb{P}^1} \xrightarrow{\text{linearly}} \underset{X}{\mathbb{P}^2}$$

$\longrightarrow$ transverse

$\longrightarrow$ double intersection

$$C_V X = N_V X \qquad \mathscr{I}_{\mathbb{P}^1}/\mathscr{I}_{\mathbb{P}^1}^2 = \mathcal{O}(-1)$$

sections of $\begin{array}{c} N_{\mathbb{P}^1}\mathbb{P}^2 \\ \downarrow \\ \mathbb{P}^1 \end{array}$ correspond to elts of $\mathcal{O}(1)$

$\mathbb{P}^2$ coords x, y, z

$\mathbb{P}^1$ $y=0$ $\qquad$ $\mathbb{P}^1$ $x=0$

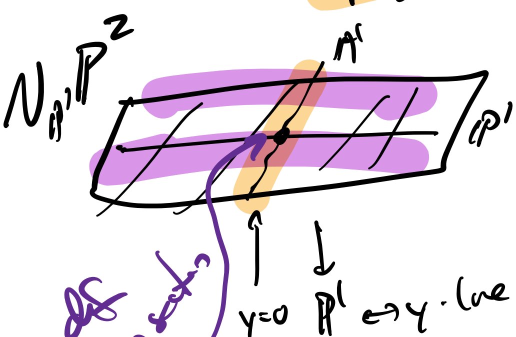
scheme theoretic $\cap$ = set theoretic $\cap$

hom. ideal (x,y) $\qquad$ all happening in affine $\mathbb{A}^2 \leftrightarrow z=1$

$$C_{\mathbb{P}^1_x \cap \mathbb{P}^1_y} \mathbb{P}^1_y \underset{W}{} \hookrightarrow \underset{V}{C_{\mathbb{P}^1_x} \mathbb{P}^2_x} \quad \operatorname{Spec}(\operatorname{Sym}^{\bullet} \mathcal{O}(-1)) = N_{\mathbb{P}^1_x} \mathbb{P}^2$$

$\parallel$ V W

$$C_0 \mathbb{A}^1 = \bigoplus x^n / x^{n+1}$$
$$= k[x]$$



by def $C_0\mathbb{A}^1 \cap$ zero section = pt

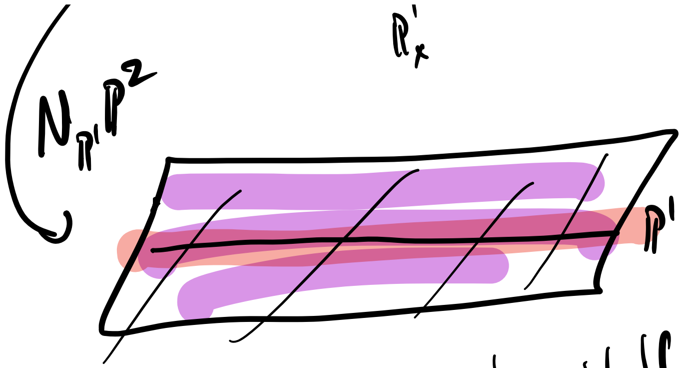
dotted lines

Q: what do we need to ensure that $\dim C_V X = \dim X$?

scheme theoretic $\cap$ $\quad \mathbb{P}^1_x \cap \mathbb{P}^1_x = \mathbb{P}^1_x$

$$C_{\mathbb{P}^1_x \cap \mathbb{P}^1_x} \mathbb{P}^1_x = \operatorname{Spec} \mathcal{O}_{\mathbb{P}^1}/0 \oplus 0/0^2 \oplus \cdots$$
$$= \operatorname{Spec} \mathcal{O}_{\mathbb{P}^1}$$

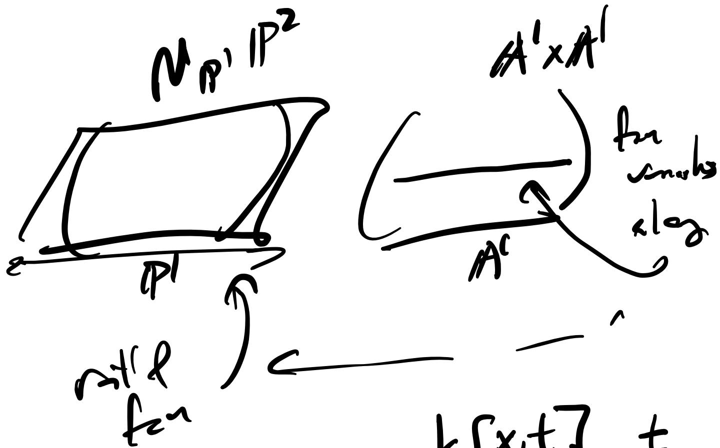
$$C_{\mathbb{P}^1 \cap \mathbb{P}^1} \mathbb{P}^1 = \mathbb{P}^1$$
$\downarrow$



look at $\mathbb{A}^1 \subset \mathbb{P}^1$ where $N_{\mathbb{P}^1}\mathbb{P}^2$ trivial
find fcn vanishes at our locus,
look at poles at ∞.

class of zero section itself.

$\mathcal{O}(1)|_{\mathbb{A}^1}$



$$\begin{array}{c} k[x,t] \quad t \\ \uparrow \\ k[x] \end{array}$$

Part 2: Intro to Chern classes.

Recall: we've defined $c_1(L)$

$$\left\{ \begin{matrix} \text{locally free rk } 1 \\ \text{sheaves} \end{matrix} \right\} \longrightarrow \text{operations on } CH$$

$c_1(L) : CH_* \to CH_{*-1}$

$[V] \longmapsto [L|_V]$

generally Cartier div in class $L|_V$

$\mathcal{E} = L_1 \oplus L_2$

$\mathcal{E}$ higher rank series of classes

$c_1 \mathcal{E} \quad c_2 \mathcal{E} \; \ldots$

$c_0 \mathcal{E} = 1$

define these so that, if we consider for $\mathcal{E}$

$c_i \mathcal{E} : CH_* \to CH_{*-i}$

$$c_t(\mathcal{E}) = 1 + c_1 \mathcal{E} + c_2 \mathcal{E} + \ldots$$
$$= 1 + c_1 \mathcal{E} t + c_2 \mathcal{E} t^2 + \ldots$$

then for $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$

$$c_t(\mathcal{E}) = c_t(\mathcal{E}_1)\, c_t(\mathcal{E}_2)$$

and $c_t(L) = 1 + c_1(L)$ — L a line bundle

i.e. $1 = c_0(L)$, higher ones are 0.

$$c_1\varepsilon_1 + c_1\varepsilon_2 = c_1(\varepsilon_1+\varepsilon_2)$$

$$(1 + c_1\varepsilon_1 + c_2\varepsilon_2 \cdots)(1 + c_1\varepsilon_2 + c_2\varepsilon_2 + \cdots)$$

$$= 1 + (c_1\varepsilon_1 + c_1\varepsilon_2) + \cdots$$