As usual, today we finish the proof of MS.

Recall: Ingredients:

H90 K_2

E/F cyclic deg p $Gal(E/F) = \langle \sigma \rangle$

$K_2(E) \xrightarrow{\sigma - 1} K_2(E) \xrightarrow{N_{E/F}} K_2(F)$ Claim: exact

H90 $\Rightarrow$ MS $\quad K_2(F)/nK_2(F) \xrightarrow{\sim} H^2(F, \mu_n^{\otimes 2})$

char $F \nmid n$

Last time: $\quad V(L) = \dfrac{\ker(N_{E \otimes L / L})}{\mathrm{im}(\sigma - 1)}$

L/F field ext.

Showed (2 lectures ago) if F prime to p closed & $N_{E/F}(E^*) = F^*$ then $V(F) = 0$ (H90 true)

Showed last time: if L/F prime to p $\Rightarrow$ $V(F) \hookrightarrow V(L)$

Stated: given $b \in F^*$ and if $E = F(\sqrt[p]{a})$ and $SB((a,b)_p)$ = Severi-Brauer variety for $(a,b)_p$

if we let $F_b = ff(SB((a,b)_\rho))$
then $b \in N_{E\otimes F_b/F_b}((E\otimes F_b)^\times)$ } classical division algebra thy.

<u>Outline</u>: <u>Assume</u> $V(F) \hookrightarrow V(F_b)$
we constructed "Merkurjev-Tom" by inductively passing to all ff's of $V(F_b)$'s simultaneously
i.e. all prime to p exts.

$\rightsquigarrow$ reduce to case F prime to p closed
i.e. $N_{E/F}$ surjective to $F^\times$

<u>Missing step</u>: $V(F) \hookrightarrow V(F_b)$
For today $X = SB((a,b)_\rho)$ $\quad F_b = F(X)$

$$\begin{array}{ccccc} K_2(E(X)) & \xrightarrow{\sigma-1} & K_2(E(X)) & \xrightarrow{N} & K_2(F(X)) \\ \uparrow & & \uparrow & & \uparrow \\ K_2(E) & \xrightarrow{\sigma-1} & K_2(E) & \xrightarrow{N} & K_2(F) \end{array}$$

suppose $u \in K_2(E)$
s.t. $N(u) = 0$
and $u_{E(X)} = (\sigma-1)(v)$
$v \in K_2(E(X))$

<u>WTS</u>: $u = (\sigma-1)(v')$
$v' \in K_2(E)$

$V(E) = 0$ (future agenda) ?

$K_2(E\otimes E) \to K_2(E\otimes E) \to K_2(E)$

$E \otimes E = \prod^{p} E$

$K_2(\prod E) = \prod K_2(E)$

as in J suppose $\sigma(v) - v = u_{E(X)}$

Consider the BGQ ss's

$$\begin{tikzcd} H^p(X, \mathcal{K}_q) \arrow[r, Rightarrow] \arrow[d] & K_{*}(X) \arrow[d] \\ H^p(X_E, \mathcal{K}_q) \arrow[r, Rightarrow] & K_{*}(X_E) \end{tikzcd}$$

look at E_1-page

$$\begin{tikzcd} & u_{E(X)} \in K_2(E(X)) \arrow[r, "d_1"] & \coprod_{x \in X_E^{(1)}} E(x)^* = K_1(E(x)) \arrow[r, "d_v"] & \coprod_{x \in X_E^{(2)}} \mathbb{Z}\cdot x \arrow[r] & 0 \\ 0 \arrow[r] & K_2(F(X)) \arrow[r] \arrow[u] & \coprod F(x)^* \arrow[r] \arrow[u] & \coprod \mathbb{Z}x \arrow[r] \arrow[u] & 0 \\ K_2(X_E) \arrow[uur, bend left, "\text{edge map of ss.}"] \end{tikzcd}$$

$v \mapsto d_1 v$, $w \mapsto d_1 v$

$K_2(X_E)$
$|$
$K_2(E) \ni u$

$Nu = 0$

$u \in K_2(E)$

$\overline{u_{E(X)}} = 0$ in $V(F(X))$

means $u_{E(X)} = \sigma v - v$
$v \in K_2(E(X))$

$\operatorname{Spec} E(X) \to X_E \to \operatorname{Spec} E \ni u$

$$\underbrace{K_2(X_E) \longrightarrow K_2(E(X)) \longrightarrow \coprod K_1(E(x))}_{\text{complex}}$$

$K_2(E) \nearrow K_2(X_E)$

$\Rightarrow d_1(u_{E(X)}) = 0$

$$\Rightarrow \quad \begin{array}{ccc} u_{E(x)} & = \sigma(v) - v \\ d_1 \downarrow & \downarrow d_1 \\ 0 & \sigma(d_1 v) = d_1(v) \end{array}$$

(Galois action is comp. w/ SS. map)

$d_1(v) \in \coprod_{x \in X_E^{(1)}} E(x)^*$

$d_1(v)$ σ-invariant $\Rightarrow$ same unpacking

$d_1(v) \in \operatorname{im} \coprod_{x \in X^{(1)}} F(x)^*$

$\Rightarrow d_1 v = w_E$

$w \in \coprod_{x \in X^{(1)}} F(x)^*$

in fact: since $d_1 d_1 v = 0$

$$d_1(w_E) = 0$$
$$\shortparallel$$
$$(d_1 w)_E \qquad d_1 w$$

$$\begin{tikzcd}
w_E \arrow[rr, bend left, mapsto] & & 0 \\
& \coprod_{x \in X_E^{(1)}} K_1(E(x)) \arrow[r, "d_1"] & \coprod_{x \in X_E^{(2)}} K_0(E(x)) & 0 \\
K_2(F(X)) \arrow[r] & \coprod_{x \in X^{(1)}} K_1(F(x)) \arrow[r] \arrow[u] & \coprod_{x \in X^{(2)}} K_0(F(x)) \arrow[u, "\text{injective}"'] \\
w \in \arrow[rr, bend right, mapsto] & & \;
\end{tikzcd}$$

so w represents a class in $H^1(X_{Zar}, \mathcal{K}_2)$ (by Gersten conj)

Claim: $H^1(X, \mathcal{K}_2) \longrightarrow H^1(X_E, \mathcal{K}_2)$ is injective

(crux thm)

but $\bar{w} \in H^1(X, \mathcal{K}_2) \mapsto 0$ in $H^1(X_E, \mathcal{K}_2)$

$$\Rightarrow \bar{w} = 0 \Rightarrow w = d_1(v')$$

$v' \in K_2(F(X))$

$$u_{E(X)} = \sigma(v) - v$$

$$= \sigma(v - v'_E) - (v - v'_E)$$

and $d_1(v - v'_E) = d_1 v - w_E$

$= 0$ by choice of w

$$\cancel{\Rightarrow d_1(\sigma(v - v'_E)) = \sigma(d_1(v - v'_E)) = 0}$$

$$\cancel{\Rightarrow d_1(u_{E(X)}) = 0}$$

$d_1(v - v'_E) = 0$ $\qquad v - v'_E \in K_2(E(X))$

$$0 \to K_2(E(X)) \longrightarrow \coprod K_1(\ ) \ - \ -$$

$$v - v'_E \longmapsto$$

$v - v'_E$ "$\in$" $H^0(X_E, \mathcal{K}_2)$ $\overset{\text{claim}}{=}$ $K_2(E)$

$$\begin{tikzcd}
\underline{\text{Claim 2}}: \quad H^0(X_E, K_2) & & K_2(E) \arrow[ll, "\sim"'] \arrow[dl, "\text{pullback}"] \\
& K_2(X_E) \arrow[ul, "\text{eg}"] &
\end{tikzcd}$$

$X_E = SB((a,b)_\ell)_E \qquad E = F(\sqrt[p]{a})$

$= SB((a,b)_\ell \otimes_F E) = SB(M_p(E))$

$= \mathbb{P}^{p-1}_E$

i.e. $\exists\, \tilde{v} \in K_2(E)$ s.t. $\tilde{v}_{E(x)} = v - v'_E$

$$((\sigma-1)\tilde{v})_{E(x)} = (\sigma-1)(v - v'_E) = (\sigma-1)v = u_{E(x)}$$

$$\begin{tikzcd}
((\sigma-1)\tilde{v})_{E(x)} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & u_{E(x)} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
H^0(X_E, K_2) = K_2(E) \arrow[d, hook] & H^0(X_E, K_2) \arrow[dl] \\
K_2(E(x)) &
\end{tikzcd}$$

$$\begin{tikzcd}
(\sigma-1)\tilde{v} - u \in K_2(E) \arrow[d, mapsto] \arrow[r, "\sim"] & H^0(K_2) \arrow[d, hook] \\
0 & K_2(E(x))
\end{tikzcd}$$