Plan: Hilbert 90 for K_2

Recall: this says if E/F cyclic degree
w/ $Gal(E/F) = \langle\sigma\rangle$ then we have an
sequence

$$K_2(E) \xrightarrow{\sigma - 1} K_2(E) \xrightarrow{N_{E/F}} K_2$$

Prove this backwards

Proof sketch:

- reduce to the case that F is prime-
  (i.e. $\forall$ L/F finite field ext, [Li
- and to the case $N_{E/F}E^* = F^*$

all the work

- In this case, explicitly finish by constructing an inverse map

$$\begin{tikzcd} \dfrac{K_2(E)}{\text{im}(\sigma-1)} \arrow[r, shift right] & K_2(F) \arrow[l, bend right] \end{tikzcd}$$

to construct inverse, recall $K_2(F) = \frac{F^*\otimes}{\langle a\otimes}$

$$\begin{tikzcd}
K_2(F) \arrow[r, "\sim"] & \\
F^*\otimes F^* \arrow[r] & \frac{K_2(E)}{\mathrm{in}(\sigma-1)} \\
a\otimes b \arrow[r, mapsto] & \{\alpha, b\}
\end{tikzcd}$$

$\alpha \in E^*$ s.t. $N_{E/F}\alpha = a$

well defined since if α' satisf

$N_{E/F}\alpha'$

$\Rightarrow N(\alpha^{-1}\alpha') = 1$

$\Rightarrow \alpha^{-1}\alpha' = \sigma(c)/c$

$\alpha' = \sigma(c)/c\,\alpha$

$\{\alpha', b\} = \{\sigma(c)/c\,\alpha, b\}$

$= \{\sigma(c)/c, b\} + \{\alpha, b$

$= \{\sigma(c), b\} - \{c, b\}$

$c \in E^*$

$(\{c,b\}) - \{c,b\}'$

$b \in F^{\times}$ $\quad = \sigma(\{c, b\})$

$$\boxed{= (\sigma - 1)\{c, b\}} + \{a, b$$

same in $\dfrac{K_2(E)}{\mathrm{im}(\sigma - 1)}$

$E_i = F(\alpha_i) \otimes_F E$

$$\begin{tikzcd} & E_i \arrow[dr, dash] \arrow[dl, dash, "\sigma/p"'] & \\ F_i = F(\alpha_i) \arrow[dr, dash] & & E \arrow[dl, dash, "/\sigma"] \\ & F & \end{tikzcd}$$

to finish, WTS
$$\begin{tikzcd} a \otimes (1-a) \arrow[r] \arrow[dr] & 0 \\ & \{\alpha, 1-a\} \end{tikzcd}$$

$N\alpha = a.$ $\quad \alpha \in E^{\times}$

consider $T^p - a = \prod p_i(T)^{n_i}$ $\quad$ = irred or 1 $\quad$ α_i roo

α root $\quad$ irred factorization in $F[T]$ $\quad$ $F_i = F$

$E_i = E$

$$1 - a = \prod p_i(1)^{n_i}$$

$$= \prod (1 - \alpha_i)^{n_i}$$

$= \prod N_{E_i/E}(1-\alpha_i)$

$\longrightarrow$ ☆

in E_i, $p_i(T) = N_{E_i/E}(T-\alpha_i)$

$$\{\alpha, 1-a\} = \{\alpha, \prod N_{E_i/E}(1-\alpha_i)^{n_i}\}$$

$$= \sum n_i \{\alpha, N_{E_i/E}(1-\alpha_i$$

proj. formula $$= \sum n_i N_{E_i/E}\{\alpha, (1-\alpha_i)$$

$$= \sum n_i N_{E_i/E}\{$$

$$= \sum n_i N_{E_i/E}\{a\alpha_i^{-1}, 1-\alpha_i$$

$N_{E/F}\alpha$

Claim: $N_{E_i/F_i}\alpha = N_{E_i/F_i}\alpha_i$ (check ~~when head~~

" $\alpha_i^p = a$ sme α_i r

$$\Rightarrow N_{E_i/F_i} a\alpha_i^{-1} = 1 \Rightarrow \alpha = \sigma\beta_i/\beta_i \, \alpha_i \quad H9$$

if L/F given by adj root β of $p(T)$

then $N_{L/F}(T-\beta) = p(T)$

$\nearrow$ β root coeff in F $\xleftrightarrow[\text{same}]{}$ dg $\qquad$ $\uparrow$ β root irred

$$\{\alpha, 1-\alpha\} = \sum n_i N_{E_i/E}\{\alpha\alpha_i^{-1}, 1-\alpha_i\}$$

$$= \sum n_i N_{E_i/E}\{\frac{\sigma\beta_i}{\beta_i}, 1-\alpha_i$$

$$= \sum n_i N_{E_i/E}\Big(\sigma\{\beta_i, 1-\alpha_i\}$$

$$- \{\beta_i, 1$$

$$= \sigma\Big(\sum n_i N_{E_i/E}\{\beta_i, 1-\alpha_i\}$$

$$- \Big(\sum n_i N_{E_i/E}\{\beta_i, 1$$

$$(\sigma - 1)\Big(\qquad \Big)$$

$\in (\sigma-1) K_2 E$.

Recap: to show exact

$$\begin{tikzcd}
K_2(E) \arrow[r, "\sigma-1"] & K_2(E) \arrow[r, "N"] & K_2(F)
\end{tikzcd}$$

$$\begin{tikzcd}
\frac{K_2(E)}{\sigma-1} \arrow[r, hook, "\tilde{N}", "\sim"'] & K_2(F) \arrow[l, bend left] \\
\{\alpha, b\} & \{a, b\} \arrow[l, bend left]
\end{tikzcd}$$

$$N_{E/F}\alpha = a$$

other fact: Bass/Tate: $K_2(E)$ gen

elements of

$\{\alpha, b\}$ α

shows surjectivity.

Sketch of reduction

$$\begin{tikzcd}
K_2(E) \arrow[r, "\sigma-1"] \arrow[d] & K_2(E) \arrow[r, "N"] \arrow[d] & K \\
K_2(E\otimes_F L) \arrow[r, "\sigma-1"] & K_2(E\otimes_F L) \arrow[r, "N_L"] & K
\end{tikzcd}$$

consider L/F linearly disjoint from E/F

$\uparrow$

Define: $V(L) = \frac{\ker N_L}{\operatorname{im}(\sigma-1)}$

WTS $V(F) = 0$

if L/F finite, prime to p

L

wrt

F

$$\begin{tikzcd}
K_2(E) \arrow[r, "\sigma-1"] \arrow[d] & K_2(E) \arrow[r, "N"] \arrow[d] & K_2(F) \arrow[d] \\
K_2(E\otimes_F L) \arrow[r, "\sigma-1"] & K_2(E\otimes_F L) \arrow[r, "N_L"] & K_2(L)
\end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d] \arrow[d, shift left=6, "N_{E\otimes L/E}"] & {} \arrow[d] \arrow[d, shift left=6, "N_{E\otimes L/E}"] & {} \arrow[d, "N"] \\ K_2(E) \arrow[r, "\sigma-1"] & K_2(E) \arrow[r, "N_L"] & K_2(F \end{tikzcd}$$

commutes

observations: $V(L) = \dfrac{\ker N_L}{(1-\sigma)}$

Claim: $V(E) = 0$ (Later)

Claim: $N_{L\otimes E/E} : K_2(L\otimes E) \longrightarrow K_2(E$

induces a map $V(L) \longrightarrow V(F$

$$\Longrightarrow \quad \begin{tikzcd} V(F) \arrow[r] \arrow[rr, bend right, "\cdot {[E:F}"'] & V(E) \arrow[r, "iso"] & V \end{tikzcd}$$

$\Rightarrow V(F)$ (and $V(L)$,

are all <u>p-torsion</u>.

$\Rightarrow$ · $[L:F]$ is an isom. on $V(F)$

" l prime to p. (l invertible mod

$\Rightarrow$

$$\begin{tikzcd}
V(F) \arrow[r, "\text{injective.}"'] \arrow[rr, bend right, "{\text{isom} \atop [L:F] \text{ prime to } p}"'] & V(L) \arrow[r] & V(F
\end{tikzcd}$$

to show $V(F)=0$ suffices to show $V(L)$

L/F prime to p.

For surj of norm.