

Plan: Hilbert 90 for K_2

Recall: this says if E/F cyclic degree n
w/ $\text{Gal}(E/F) = \langle \sigma \rangle$ then we have an
sequence

$$K_2(E) \xrightarrow{\sigma-1} K_2(E) \xrightarrow{N_{E/F}} K_2$$

Prove this backwards

Proof sketch:

- all the work \rightarrow
- reduce to the case that F is prime!
i.e. $\forall L/F$ finite field ext, $[L:F] = n$
 - and to the case $N_{E/F} E^* = F^*$
 - In this case, explicitly finish by constructing an inverse map

$$\frac{K_2(E)}{\text{im}(\sigma-1)} \xrightarrow{\quad} K_2(F)$$

to construct inv, recall $K_2(F) = \frac{F^*}{\langle a \otimes b \rangle}$

$$\begin{array}{ccc}
 K_2(F) & \xrightarrow{\sim} & K_2(E) \\
 F^* \otimes F^* & \xrightarrow{\quad} & \frac{F^* \otimes F^*}{\langle a \otimes b \rangle} \\
 a \otimes b & \xrightarrow{\quad} &
 \end{array}$$

$$\{a, b\}$$

$$a \in F^* \text{ s.t. } N_{E/F} a = a$$

well defined since if a' s.t. $N_{E/F} a' = a$

$$\Rightarrow N(a^{-1}a') = 1$$

$$\Rightarrow a^{-1}a' = \sigma(c)/c$$

$$a' = \sigma(c)/c a$$

$$\{a', b\} = \left\{ \frac{\sigma(c)}{c} a, b \right\}$$

$$= \left\{ \frac{\sigma(c)}{c}, b \right\} + \{a, b\}$$

$$= \{ \sigma(c), b \} - \{c, b\}$$

$$= \{ \sigma(c), b \} - \{c, b\}$$

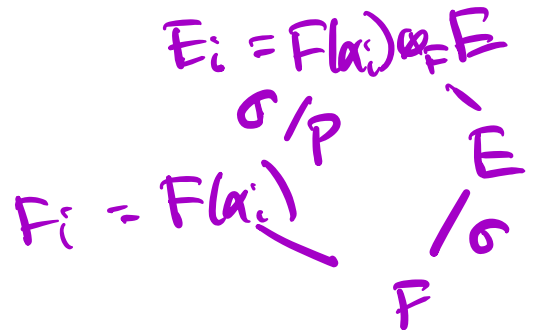
$$c \in B^*$$

$b \in F^*$

$= \sigma(\{c, b\})$

$\frac{(0-1)\{c, b\} + \{a, b\}}$

same in $\frac{K_2(E)}{m(\sigma-1)}$



to finish, WTS

$a \& (1-a) \rightarrow 0$

$\searrow \{a, 1-a\}$

$N\alpha = a, \alpha \in E^*$

consider $T^p - a = \prod p_i(T)^{n_i}$ = irred or 1

α root

irred factors in $F(T)$

α_i root

$F_i = F$

$E_i = E$

$1-a = \prod p_i(1)^{n_i}$

$\dots (1-a)^{n_i}$

$$= \prod N_{E_i/E} (1-\alpha_i)$$

→ ~~★~~ in E_i , $p_i(T) = N_{E_i/E} (T-\alpha_i)$

$$\{\alpha, 1-\alpha\} = \{\alpha, \prod N_{E_i/E} (1-\alpha_i)^{n_i}\}$$

$$= \sum n_i \{\alpha, N_{E_i/E} (1-\alpha_i)\}$$

proj. formula

$$= \sum n_i N_{E_i/E} \{\alpha, (1-\alpha_i)\}$$

$$= \sum n_i N_{E_i/E} \{\alpha, 1-\alpha_i\}$$

$$= \sum n_i N_{E_i/E} \{\alpha \alpha_i^{-1}, 1-\alpha_i\}$$

Claim: $N_{E_i/F_i} \alpha = N_{E_i/F_i} \alpha_i$ (check when head)

$$= \alpha_i^p = \alpha \text{ since } \alpha_i \in K$$

$$\Rightarrow N_{E_i/F_i} \alpha \alpha_i^{-1} = 1$$

$$\Rightarrow \alpha = \frac{\sigma_i}{\rho_i} \alpha_i \quad \text{HA}$$

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~~if~~ L/F given by adj root β of $p(T)$
 Then $N_{L/F}(T-\beta) = p(T)$
 β root call in F \longleftrightarrow same β root in rmd

$$\begin{aligned}
 \zeta_{\alpha, 1-\alpha} &= \sum n_i N_{E_i/E} \zeta_{\alpha \alpha_i^{-1}, 1-\alpha_i} \\
 &= \sum n_i N_{E_i/E} \zeta_{\frac{\sigma \beta_i}{\beta_i}, 1-\alpha_i} \\
 &= \sum n_i N_{E_i/E} \left(\sigma \zeta_{\beta_i, 1-\alpha_i} - \zeta_{\beta_i, 1} \right) \\
 &= \sigma \left(\sum n_i N_{E_i/E} \zeta_{\beta_i, 1-\alpha_i} \right) - \left(\sum n_i N_{E_i/E} \zeta_{\beta_i, 1} \right) \\
 &= (\sigma - 1) (\quad)
 \end{aligned}$$

$$\epsilon(\sigma^{-1}) K_2 E.$$

Recap: to show exact

$$K_2(E) \xrightarrow{\sigma^{-1}} K_2(E) \xrightarrow{N} K_2(F)$$

$$\frac{K_2(E)}{\sigma^{-1}} \xrightarrow{N} K_2(F)$$

$\leftarrow \quad \leftarrow$
 $\{a, b\} \quad \{a, b\}$

$$N\alpha = a$$

E/F

other fact: Bass (Tate): $K_2(E)$ gen. elements - f. $\{a, b\}$ α shows surjectivity.

Sketch of reduction

consider L/F (notly direct from E/F)

$$\begin{array}{ccccc}
 K_2(E) & \xrightarrow{\sigma^{-1}} & K_2(E) & \xrightarrow{N} & K \\
 \downarrow & & \downarrow & & \downarrow \\
 K_2(E \otimes_P L) & \xrightarrow{\sigma^{-1}} & K_2(E \otimes_{P \otimes L}) & \xrightarrow{N_L} & K
 \end{array}$$

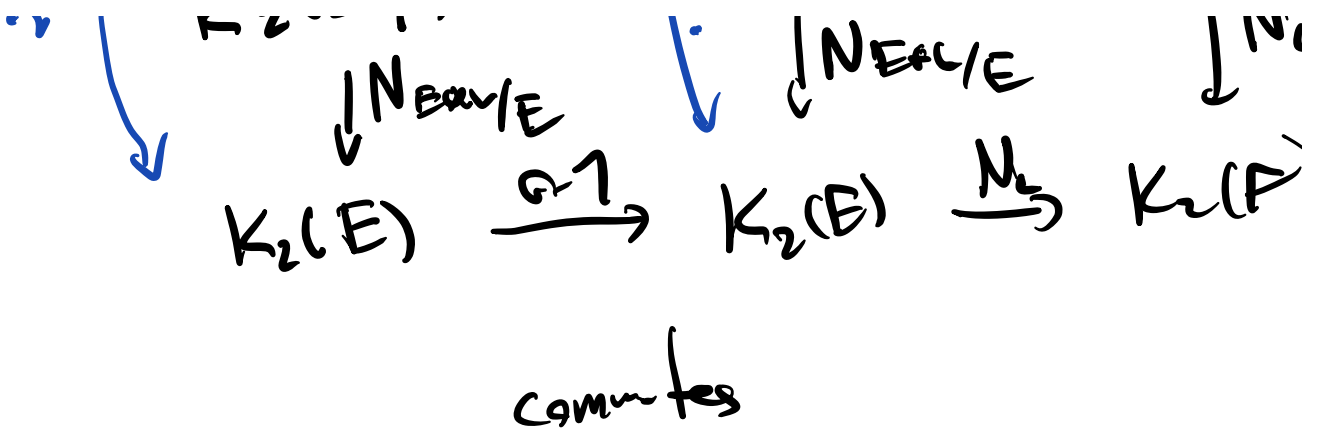
Define: $V(L) = \frac{\text{ker } N_L}{\text{im}(\sigma^{-1})}$

WTS $V(F) = 0$

if L/F finite, pure to P \hookrightarrow not P'

$V(F)$

$$\begin{array}{ccccc}
 K_2(E) & \xrightarrow{\sigma^{-1}} & K_2(E) & \xrightarrow{N} & K_2(F) \\
 \downarrow & & \downarrow & & \downarrow \\
 V_0(E \otimes_P L) & \xrightarrow{\sigma^{-1}} & K_2(E \otimes_{P \otimes L}) & \xrightarrow{N_L} & K_2(L)
 \end{array}$$



observation $V(L) = \frac{kr N_L}{(1-\sigma)}$

Claim: $V(E) = 0$ (Later)

Claim: $N_{L \otimes E/E}: K_2(L \otimes E) \rightarrow K_2(E)$

induces a map $V(L) \xrightarrow{\cong} V(F)$

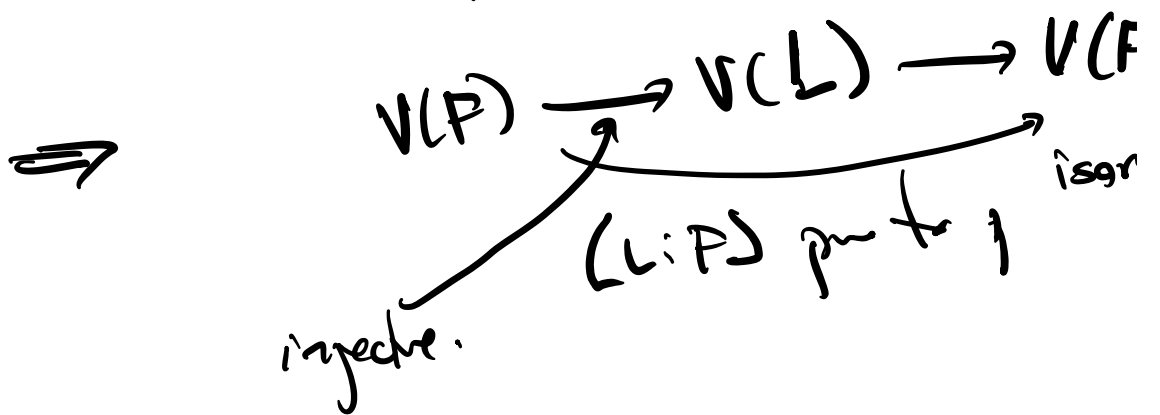
$\Rightarrow V(F) \rightarrow V(E) \rightarrow V$

$\cdot [E:F]$

...

$\Rightarrow V(F)$ and $V(L)$,
are all p-prime.

$\Rightarrow \cdot [L:F]$ is an isom. on $V(F)$
"l prime to p. (l divisible mod."



to show $V(F) = 0$ suffices to show $V(L)$
 L/F prime to p .

For surj of non.