Plan: Merkurjev-Suslin Theorem

Lots of ingredients

- Hilbert's Theorem 90 for K_2
- Adjustment of above for K_2/pK_2 & extension
- Relatively short argument afterwards

Hilbert's theorem 90

Classical statement: if E/L is a cyclic extension w/ group $\langle\sigma\rangle$

then $N_{E/L}(a) = 1$ iff $a = \sigma(b)/b$ some $b \in E^*$

i.e. $\exists$ exact sequence

$$E^* \xrightarrow{\sigma-1} E^* \xrightarrow{N} F^*$$

ab-gp A w/ G-action, get a $\mathbb{Z}[G]$ module

extends to left

$$F^* \longrightarrow E^* \xrightarrow{\sigma-1} E^* \xrightarrow{N} F^*$$

H90 for K_2 says E/F cyclic gp $\langle\sigma\rangle$

have an exact sequence

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N_{E/F}} K_2(F)$$

what are these maps?

- how does $Gal(E/F)$ act on $K_2(E)$
- what is $N_{E/F}$?

$N_{E/F}$ given by π_* $\quad \operatorname{Spec} E \xrightarrow{\pi} \operatorname{Spec} F$ proper.

Action of $\operatorname{Aut}(R)$ on $\operatorname{Mod}(R)$

i.e. $\sigma \in \operatorname{Aut}(R)$ induce $\operatorname{Mod}(R) \circlearrowleft$

if M an R-module: ${}^\sigma M = M$ as Ab. gps $\{{}^\sigma(m) \mid m \in M\}$

but w/ $r \cdot {}^\sigma(m)$

$\quad " \sigma(\sigma^{-1}(r)\cdot m)$

$f: M \longrightarrow N$

$\sigma(f) \equiv " \sigma f \sigma^{-1} " : {}^\sigma M \longrightarrow {}^\sigma N$

Recall $K_1(E) = E^\times$ $\circlearrowleft$ $\quad K_2(E)$ gen by $\{a,b\}$ $\circlearrowleft$

$\overset{\cup}{\text{standard}}$ $\sigma(\{a,b\}) = \{\sigma(a),\sigma(b)\}$

Statement of Norm residue isom thm / BK in deg 2

is that the map $(\operatorname{char} F \nmid n)$

$$\underset{\substack{\| \\ F^\times}}{K_1(F)} \longrightarrow H^1(F,\mu_n) = F^\times/(F^\times)^n$$

$$\| \quad H^1(\underbrace{\mathrm{Gal}(F^s/F)}_{G}, \mu_n(F^s))$$

$$1 \to \mu_n(F^s) \to (F^s)^\times \xrightarrow{n} (F^s)^\times \to 1$$

$$H^0(G, F^{s\times}) \xrightarrow{n} H^0(G, F^{s\times}) \to H^1(G, \mu_n(F^s))$$

$$(F^{s\times})^G = F^\times \longrightarrow F^{\downarrow\times} \qquad \underbrace{H^1(G, F^{s\times})}_{0 \text{ "H90"}} \to$$

$$\{a\} \longrightarrow (a)_n$$

$$F^\times = K_1(F) \longrightarrow H^1(F,\mu_n) = F^\times/(F^\times)^n$$

$$K_2(F) \longrightarrow H^2(F, \mu_n^{\otimes 2})$$

$$\{a,b\} \longrightarrow (a) \cup (b) \equiv (a,b)$$

gen by $\{a,b\} = \underset{\substack{\in \\ K_1(F)}}{\{a\}} \cdot \underset{\substack{\in \\ K_1(F)}}{\{b\}}$

$$\mu_n^{\otimes 2} = \mu_n \otimes_{\mathbb{Z}} \mu_n$$

$\rho \in \mu_n(E)$ $\quad \sigma \in \mathrm{Gal}(E/F)$

$\mu_3(F^s)$ $\quad\quad$ $\mathbb{Z}/3 \cong \mu_3^{\otimes 2}$ $\quad$ $1, \rho, \rho^2$

$1 \otimes 1 = 0$, $\rho \otimes 1 = 0$, $\rho \otimes \rho$

$1 \otimes$

$\mu_3^{\otimes 2}(F^s) \cong \mathbb{Z}/3\mathbb{Z}$ $\quad\quad$ $\rho \otimes \rho^2$, $\rho^2 \otimes \rho$, $= \rho^2 \otimes \rho^2$

$$H^2(G(F^s/F), \mu_n(F^s)) \quad \rho \otimes \rho = \rho^4 \otimes \rho$$

<u>FACT</u>: $H^2(F, \mu_n) = Br(F)[n]$

$$1 \to \mu_n(F^s) \to (F^s)^* \xrightarrow{n} (F^s)^* \to 1$$

$$\text{H90} \quad 0 \to H^2(G(F^s/F), \mu_n(F^s)) \to H^2(G(F^s/F), F^{s*}) = Br(F)$$

$H^2(F, \mu_n) = H^2(G(F^s/F), \mu_n(F^s))$ $\quad\quad$ $\downarrow \cdot n$

$H^2(G \cdots) = Br(F)$

$$H^2(F, \mu_n) \not\cong H^2(F, \mu_n^{\otimes 2})$$

but it is if $Gal(F^s/F)$ acts trivially on μ_n.

Theorem (Merkurjev–Suslin) if $\operatorname{char} F \nmid n$

$$K_2(F) \longrightarrow H^2(F, \mu_n^{\otimes 2})$$

induces an isomorphism $R_{n,F} : K_2(F)/n \to H^2(F, \mu_n^{\otimes 2})$

"norm residue map"

"Galois symbol"

Series of standard reductions:

- reduce to the case n prime $= p$
- reduce to the case all finite extensions of F are p-powers.

(In "bad" characteristic (Gille/Szamuely)

$$K_2(F) \longrightarrow \Omega^2_F = \Lambda^2_F \Omega^1_F \rightsquigarrow H^2(F, \mu_\ell^{\otimes 2})$$

$$\{a, b\} \longmapsto \frac{da}{a} \wedge \frac{db}{b}$$

)

Summarize bits:

- First reduce to $n=p$ prime
- F is prime-to-p closed. (restriction-corestriction)

1 $\boxed{H90}$ (great! because now WLOG, $\mu_p \subset F$ refined $H90/p$ $E = F(\sqrt[p]{b})$

★ K-thy if E/F cyclic deg p $\langle\sigma\rangle$

$$\left[\, K_2(F)/p \times K_2(E)/p \xrightarrow{i \times (\sigma - 1)} K_2(E)/p \xrightarrow{N} K_2(F)/p \,\right] ★$$

2

also

$$K_1(F)/p \xrightarrow{\cup\{b\}} K_2(F)/p \longrightarrow K_2(E)/p$$

2'

final argument (not quite as hard) $(F \supset \mu_p)$

if E/F deg p and $R_{p,F}$ is injective

then

3

$$\begin{array}{ccc} K_2(F)/p & \xrightarrow{R_{p,F}} & Br(F)[p] \\ \downarrow & & \downarrow \\ K_2(E)/p & \xrightarrow[R_{p,E}]{} & Br(E)[p] \end{array}$$

is a pullback $\square$.

$1, 2, 2', 3 \Rightarrow MS.$

Step 1: $R_{p,F}$ is injective.

Suppose $x = \sum_{i=1}^{m} \{a_i, b_i\}$ s.t. $R_{p,F}(x) = 0$

induct on m

wLOG, $a_m \notin (F^\times)^p$ $\quad \{a_m^p, b_m\} = p\,\cancel{\{a_m, b_m\}}$

Let $E = F(\sqrt[p]{a_m})$

$x_E \in K_2(E)/p$ $\quad x_E = \sum_{i=1}^{m-1} \{a_i, b_i\}$

$R_{p,E}(x_E) = 0 \Rightarrow x_E = 0$

$\Rightarrow x = \{a_m, y\} \quad y \in F^\times$

it suffices to check the case $x = \{a, b\}$

Single symbol, $\{a,b\}$, we'll do it by hand. is a single symbol

$R_{p,E}(\{a,b\}) = (a,b)$ "cyclic algebra"

Classical: $(a,b) = 0$ in $Br(F)$

$\Leftrightarrow b \in N_{F(\sqrt[p]{a})/F}(\beta)$ we'll need to observe (later)

"easy" $\longrightarrow \Rightarrow \{a,b\} \in pK_2(F)$

surjectivity:

choose $\alpha \in Br(F)[p]$

choose E/F min'l degree s.t. $\alpha_E = 0$

induct on $[E:F]$

Base case $[E:F] = p$

$$
\begin{tikzcd}
 & \alpha & \\
K_2(F)/p \arrow[r] \arrow[d] & Br(F)[p] \arrow[d] & \\
K_2(L)/p \arrow[r] \arrow[d] & Br(L)[p] \arrow[d] & \alpha_L \\
K_2(E)/p \arrow[r] & Br(E)[p] & 0
\end{tikzcd}
$$