Towards BGQ S.Seq.

1. More about localization sequences
2. Limits of these, stick together via exact couples

$Z \hookrightarrow X$ closed $\quad U = X \setminus Z$

$$K_n(Z) \to K_n(X) \to K_n(U) \to K_{n-1}(Z)$$

3. (?) Severi-Brauer Varieties.

Interpretation of localization sequence.

Already seen part of these $\quad X$ reg. variety $Z \subset X$ closed irred codim 1

$$\begin{tikzcd}
 & k[U]^\times \arrow[d] \arrow[r, "\mathrm{div}(f)"] & \mathbb{Z} & & & \\
K_1(X) \arrow[r] & K_1(U) \arrow[r] & K_0(Z) \arrow[r] & K_0(X) \arrow[r] & K_0(U) \arrow[r] & 0 \\
 & & \mathbb{Z}[Z] \arrow[u] \arrow[r] & \mathrm{Pic}\, X \arrow[u] \arrow[r] & \mathrm{Pic}\, U \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

pretend exactly is also.

"Z small"

in general, these maps

$$K_n(U) \to K_{n-1}(Z)$$

are "refined residue maps"

higher versions of 0's or poles.

e.g. in limit, if we localize at the gen pt. of Z

$$\mathcal{O}_X(U) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_{X,\eta} \qquad \eta \in Z \text{ gen pt.}$$

$$= \mathrm{frac}(\mathcal{O}_{X,\eta})$$

$$K_n(\underbrace{\mathrm{frac}(\mathcal{O}_{X,\eta})}_{k(X)}) \to K_{n-1}(\underbrace{\mathcal{O}_{X,\eta}/m_{X,\eta}}_{k(Z)})$$

$$\begin{array}{ccccc} K_1(\mathcal{O}_{X,\eta}) & \to & K_1(k(X)) & \to & K_0(k(Z)) \\ & & \| & & \| \\ \mathcal{O}_{X,\eta}^* & & k(X)^* & \xrightarrow[\mathrm{div}_Z \,=\, \nu_Z]{} & \mathbb{Z} \end{array}$$

Meck

$$K_1(k(X))/\ell$$

$$\mathcal{O}_{X,\eta}^*/\ell \to k(X)^*/\ell \xrightarrow{\text{name @ } \eta} \mathbb{Z}$$

cyclic exts

$$K_2(\mathcal{O}_{X,\eta}) \to K_2(k(X)) \xrightarrow{\partial_Z} K_1(k(Z))$$

// classical

gen by $\{a,b\}$ $a,b \in k(X)^*$

w/ rels $\{aa',b\} = \{a,b\} + \{a',b\}$

$\{a,bb'\} = \{a,b\} + \{a,b'\}$

$\{a,1-a\} = 0$

$$\partial_Z(\{a,b\}) = (-1)^{v(b)v(a)} \left(\frac{b^{v(a)}}{a^{v(b)}} \right) \in k(Z)^* = K_1(k(Z))$$

$$K_2(\mathcal{O}_{X,\eta})/\ell \to K_2(k(X))/\ell \to K_1(k(Z))/\ell$$

$\cong$ if $\mu_\ell \subset k$

$$Br(\mathcal{O}_{X,\eta})[\ell] \to Br(k(X))[\ell] \xrightarrow{\text{ram @ } \eta} k(Z)^* / (k(Z)^*)^\ell$$

$$\begin{array}{ccccc} K_3(\mathcal{O}_{X,\eta}) & \to & K_3(k(X)) & \to & K_2(k(Z)) \\ \uparrow & & \uparrow & & \uparrow \\ K_3^M(\mathcal{O}_{X,\eta}) & \to & K_3^M(k(X)) & \to & K_2^M(k(Z)) \\ \downarrow & & \downarrow /2 & & \downarrow /2 \\ \text{Octonion}/\mathcal{O}_{X,\eta} & \xrightarrow{?} & \text{Octonion}/k(X) & \to & Br(k(Z))[2] \end{array}$$

$\{a,b,c\}$

$H^3(k(X), \mathbb{Z}/2)$

<u>**Notation:**</u> $\mathrm{Coh}(X) \leftrightarrow \mathcal{M}(X)$

$\mathcal{M}_Z(X)$ = coh sheaves on X supp. on Z.

i.e. $\sqrt{\mathrm{supp}(\mathcal{F})} = Z$

Fact from before

$$\mathcal{M}(X) / \mathcal{M}_Z(X) \underset{eq}{\simeq} \mathcal{M}(U) \qquad U = X \setminus Z$$

$$BQ(\mathcal{M}(X)) \longrightarrow BQ(\mathcal{M}(U))$$

has homology fiber

$$BQ(\mathcal{M}_Z(X))$$

all coh sheaves in $\mathcal{M}_Z(X)$ can be resolved by sheaves killed by $\mathcal{I}_Z$

$$0 = \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F} \to 0$$

$\mathcal{F} \in \mathcal{M}_Z(X)$ $\qquad \mathcal{O}_Z$

$\mathcal{F}_i / \mathcal{F}_{i+1}$ killed by $\mathcal{I}_Z$ $\to$ mod. over $\mathcal{O}_X / \mathcal{I}_Z$

Devissage $\Rightarrow$ $K_n(\mathcal{M}_Z(X)) \simeq K_n(\mathcal{M}(Z))$

$\left(\mathcal{M}(Z) \simeq \text{subcat of } \mathcal{M}_Z(X) \text{ of modules killed by } \mathcal{I}_Z \right)$

les

$$\begin{array}{ccccc} K_n(Z) & & K_n(X) & & K_n(U) \\ \| & & \| & & \\ K_n(\mathcal{M}_Z(X)) & \to & K_n(\mathcal{M}(X)) & \to & K_n(\mathcal{M}(U)) \\ & & & & \downarrow \\ & & & & K_{n-1}(\mathcal{M}_Z(X)) \\ & & & & \| \\ & & & & K_{n-1}(Z) \end{array}$$

let $\mathcal{M}(X)^p$ = full subcat of $\mathcal{M}(X)$ of coh sheaves $\mathcal{F}$ s.t. $\mathrm{supp}(\mathcal{F})$ codim $\geq p$

ex. $\mathcal{M}(X)^1 \hookrightarrow \mathcal{M}(X)$

$\| \; \bigcup \mathcal{M}_Z(X)$ Z closed codim 1

$\mathcal{M}(X) / \mathcal{M}(X)^1$

$\| \; \text{"}\lim_{U \supset X}\text{"} \mathcal{M}(U) \overset{\text{guess}}{=} \mathcal{M}(k(X))$

$\mathcal{M}(X)^1$ = all torsion modules.

supp($\mathcal{F}$) codim 1 $\leftrightarrow$ ann($\mathcal{F}$) $\neq (0)$
$\simeq$

proper in X $\quad X$

affine $\exists f \neq 0$ f kills $\mathcal{F}$

Classic localization:

R ring $S \subset R$ mult. subset

$\mathrm{Mod}(R) \supset \mathrm{Mod}(R, S\text{-torsion})$

Serre subcat

$$\mathrm{Mod}(R) \big/ \mathrm{Mod}(R, S\text{-tors}) = \mathrm{Mod}(RS^{-1})$$

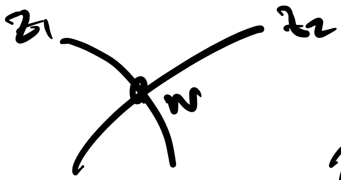
in analogy: $\mathcal{M}(X) \big/ \mathcal{M}^1(X) \underset{eq}{\simeq} \mathcal{M}(k(X))$

$$\mathcal{M}(A \times B) = \mathcal{M}(A) \times \mathcal{M}(B)$$

$$\mathcal{M}^1(X) \big/ \mathcal{M}^2(X) \to \coprod_{\substack{\eta \\ \text{codim } 1}} \mathcal{M}(\mathcal{O}_{X,\eta}) \quad \eta \text{ codim } 1$$

$$\mathcal{M}^1(X) \big/ \mathcal{M}^2(X) \xrightarrow{\sim} \coprod_{\eta} \mathcal{M}(\mathcal{O}_{X,\eta}/I)$$

$\sqrt{I} = \eta$



in quotient $/\mathcal{M}^2(X)$

$$\mathcal{M}_{Z_1\cup Z_2}(X)/\mathcal{M}_w(X) \xrightarrow{\sim} \mathcal{M}_{Z_1}(X)/\mathcal{M}_w(X) \times \mathcal{M}_{Z_2}(X)/\mathcal{M}_w(X)$$

$$\mathcal{O}_{(Z_1\cup Z_2)\setminus w} = \mathcal{O}_{(Z_1\setminus w)} \times \mathcal{O}_{(Z_2\setminus w)}$$

$$\begin{tikzcd}
K_n(\mathcal{M}^{p+1}(X)) \arrow[r] & K_n(\mathcal{M}^p(X)) \arrow[r] & \coprod_{\eta\in X^{(p)}} K_n(k(\eta)) \arrow[dll, bend left] \\
K_{n-1}(\mathcal{M}^{p+1}(X)) & &
\end{tikzcd}$$

$$\begin{tikzcd}
K_\bullet(\mathcal{M}^*(X)) \arrow[rr, "* \mapsto *-1"] & & K_\bullet(\mathcal{M}^*(X)) \arrow[dl] \\
& \coprod_{\eta\in X^{(*)}} K_\bullet(k(\eta)) \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
D^1 \arrow[rr, "\beta"] & & D^1 \arrow[dl, "\gamma"] \\
& E^1 \arrow[ul, "\alpha"] &
\end{tikzcd}$$

$$\begin{tikzcd}
D^2 \arrow[rr] & & D^2 \arrow[dl] \\
& E^2 \arrow[ul] &
\end{tikzcd}$$

$E^2 =$ Hom of $E^1 \to E^1 \to E$

$$\boxed{E_1^{p,q} = \coprod_{x \in X^{(p)}} K_{-p-q}(k(x)) \Rightarrow \begin{array}{l} G_{-p-q}(X) \\ K_{-p-q}(X) \end{array}}$$