<u>Recall:</u>

Ingredients from last time

$$\underset{\text{exact}}{\mathcal{C}} \rightsquigarrow Q\mathcal{C}$$

objects $= \mathrm{ob}(\mathcal{C})$

morphisms $\leftrightarrow$ subquotient realizations

$$\begin{array}{c} A \longrightarrow B \\ \cup \\ B' \\ \cup \\ B'' \end{array} \qquad A \xrightarrow{\sim} B'/B''$$

$$\begin{tikzcd} & B' \arrow[dl, two heads] \arrow[dr, hook] & B'' \arrow[l, tail] \\ A & & B \end{tikzcd}$$

Nerve construction

$\mathcal{C} \rightsquigarrow N\mathcal{C}$ simplicial complex

$N\mathcal{C}_0 = \mathrm{ob}(\mathcal{C})$

$N\mathcal{C}_1 =$ morphisms

$N\mathcal{C}_i = a_0 \to a_1 \to \cdots \to a_i$

Classifying space $B\mathcal{C} = |N\mathcal{C}|$

Def $K_i(\mathcal{C}) = \pi_{i+1}(BQ\mathcal{C}, [0])$

$\quad\uparrow$ exact

$0 \in ob\,\mathcal{C} = ob(Q\mathcal{C}) = (NQ\mathcal{C})_0$

Rem. $K_0(\mathcal{C}) \xrightarrow{\sim} \pi_1(BQ\mathcal{C}, [0])$

$[A] \longmapsto$ $[0] \rightarrow [A]$

$$\boxed{\begin{tikzcd} & 0 \arrow[dl, two heads, "\cong"'] \arrow[dr, tail] & \\ 0 & & A \end{tikzcd}} \quad 0 \to A$$

$$\boxed{\begin{tikzcd} & A \arrow[dl, two heads] \arrow[dr, tail, "\cong"] & \\ 0 & & A \end{tikzcd}}^{-1} \quad 0 \to A$$

Properties of B which we'll use later

$F : \mathcal{C} \to \mathcal{C}'$ functor

$\rightsquigarrow BF : B\mathcal{C} \to B\mathcal{C}'$

if $\theta : F \Rightarrow G$ nat trans then $B\theta$ is a homotopy between BF & BG

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}' \end{tikzcd}$$

$B\theta : B\mathcal{C} \times I \to B\mathcal{C}'$ homotopy between BF & BG

$\mathcal{C} \times \mathcal{I} \longrightarrow \mathcal{C}'$ functors are same as pair of functors and nat trans between them

$\mathcal{I}$ cat w/ $\underset{0}{\cdot} \to \underset{1}{\cdot}$

$B(\mathcal{D}_1 \times \mathcal{D}_2) = B\mathcal{D}_1 \times B\mathcal{D}_2$

product in cat of compactly gen spcs.

$$\left(\begin{array}{c}\text{Compactly} \\ \text{gen spcs}\end{array}\right) \rightleftarrows (\text{Top spcs})$$

U open $\Leftrightarrow$ $U \cap C$ open in C all C compact

If a functor has a (left or right) adjoint $\leadsto$ hom. equivalence.

In particular, if $\mathcal{C}$ has an initial or final object $\Rightarrow$ $B\mathcal{C}$ is contractible.

<u>Reminder</u>: If $X \xrightarrow{f} Y$ cont map, $y \in Y$, $x \longmapsto y$

define Homotopy fiber $F(f, y)$ to be the space

$\{(x,p) \in X \times \text{Paths in } Y \mid f(x) \rightsquigarrow^{p} y\}$

$\rightsquigarrow$ exact sequces in homotopy groups

$$\to \pi_{i+1}(Y,y) \to \pi_i(F(f,y),\underline{x}) \to \pi_i(X,x) \to \pi_i(Y,y) \to$$
$y = f(x)$

<u>Main interest:</u> $\mathcal{C} = \text{Coh-}\mathcal{O}_X\text{-mod}$ $\quad X$ ng. sche

Loc. free $\mathcal{O}_X$-mod

$$K_i(\mathrm{Coh}_{\mathcal{O}_X}) = K_i(X)$$
$$K_i(\mathrm{Locfree}_{\mathcal{O}_X}) = G_i(X)$$

Given X, Y Noeth scheme

$$f : X \to Y$$

$$f^* : LF(Y) \longrightarrow LF(X)$$

is an exact functr.

$\rightsquigarrow$ induced $\quad Q\,LF(Y) \longrightarrow Q\,LF(X)$

$\rightsquigarrow f^* : K_i(Y) \longrightarrow K_i(X)$

on other hand, if $f: X \to Y$ flat,

also get $f^*: \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ exact

$\rightsquigarrow f^*: G_i(Y) \to G_i(X)$

alternately, if X is Noeth regular, get $G_i(X) = K_i(X)$

since (by ex. in Hartshorne) every coh. sheaf

has a finite res by L.free sheaves

"Res. theorem"

$$K_i(LF(X)) = K_i(\mathrm{Coh}(X))$$

Pushforwards

$f: X \to Y$ proper morphism

$\mathcal{F}$ coh $\rightsquigarrow$ $f_*\mathcal{F}$ coh. $R^i f_* \mathcal{F}$ coh.

let $\mathrm{Coh}(X, f)$ = sub cat of coh sheaves $\mathcal{F}$ s.t. $R^i f_* \mathcal{F} = 0$ $i > 0$

$\Rightarrow$ $f_*: \mathrm{Coh}(X, f) \to \mathrm{Coh}(Y)$ is exact.

$\rightsquigarrow$ $f_*: K_i(\mathrm{Coh}(X, f)) \to K_i(\mathrm{Coh}(Y))$

show (if X is q. projective) that every object in $Coh(X)$ is resolvable by objects in $Coh(X,A)$

$\Rightarrow K_i(Coh(X,A)) \cong K_i(Coh\, X)$

$\Rightarrow f_*: G_i(X) \to G_i(Y)$