<u>Today</u>: Quillen's Q construction and def. of higher K-theory.

$\mathcal{C}$ = Abelian category $\quad K_i(\mathcal{C})$

(f.g. modules over rings, coh. sheaves, loc. free sheaves)

more generally $\mathcal{C}$: exact category

<u>Def of exact category</u>: $\mathcal{C}$

An exact cat is a subcat of an Abelian cat $\mathcal{A}$ which is closed under extensions - i.e.

if $0 \to M' \to M \to M'' \to 0$ exact in $\mathcal{A}$ and $M', M'' \in ob(\mathcal{C}) \Rightarrow M$ is isom. to some object in $\mathcal{C}$

given an exact cat $\mathcal{C}$, we say a morphism

$X \to Y$ is an admissible mono (write $X \rightarrowtail Y$)

if $\exists$ an exact seq.

$0 \to X \to Y \to Z \to 0$

$Y \to Z$ is an admissible epi

$(Y \twoheadrightarrow Z)$

if $\exists$ exact seq

$0 \to X \to Y \to Z \to 0.$

ex: $\mathcal{C} \overset{\text{f.g.}}{\vee}$ tors. free gps $\quad \mathcal{C} \hookrightarrow \mathcal{A} = $ Ab. gps.

$2\mathbb{Z} \to \mathbb{Z}$ non-admissible mono.

$X \to Y$ admissible mono $\Rightarrow$ cokr is torsion free

if $Y \xrightarrow{f} Z$ surj., Y, Z torsion free

Z proj $\Rightarrow Y \cong \ker f \oplus Z$

$K_i(\mathcal{C})$ four step process

1: Q construction : $Q\mathcal{C}$ new category

2: Nerve of $Q\mathcal{C}$ = simplicial set $NQ\mathcal{C}$

3: Geometric realization of $NQ\mathcal{C}$ $\quad |NQ\mathcal{C}|$

$\qquad ||$

$BQ\mathcal{C}$

4: $K_i(\mathcal{C}) = \pi_{i+1}(BQ\mathcal{C})$

Q
N
B

"today"

Serre localization in Ab cat

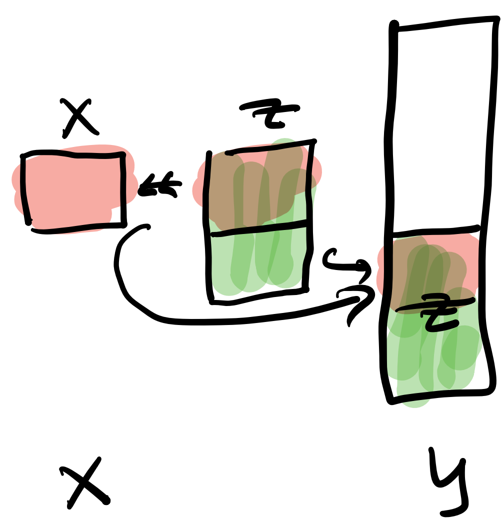
$\rightsquigarrow$

BQ ?

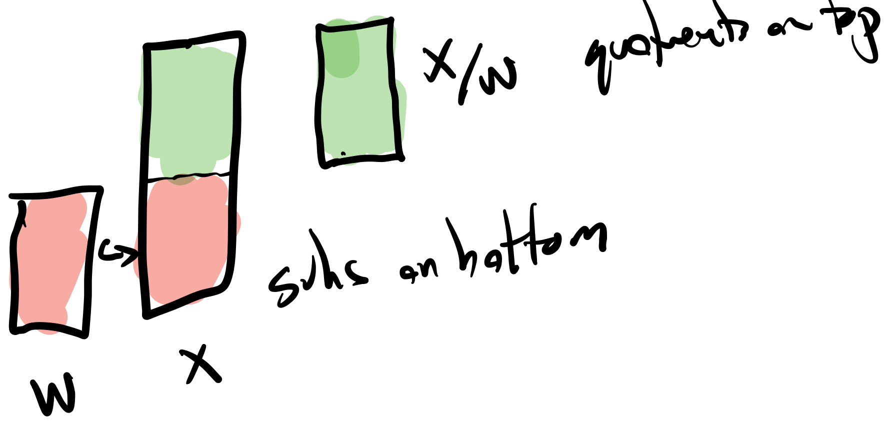
<u>Def</u>: Let $\mathcal{C}$ be an exact category.
The cat $Q\mathcal{C}$ has same objects as $\mathcal{C}$, but morphism are given as

$$\mathrm{Hom}_{Q\mathcal{C}}(X,Y) = \left\{ \begin{tikzcd} & Z \arrow[dl, two heads] \arrow[dr, hook] & \\ X & & Y \end{tikzcd} \right\} \Big/ \sim$$

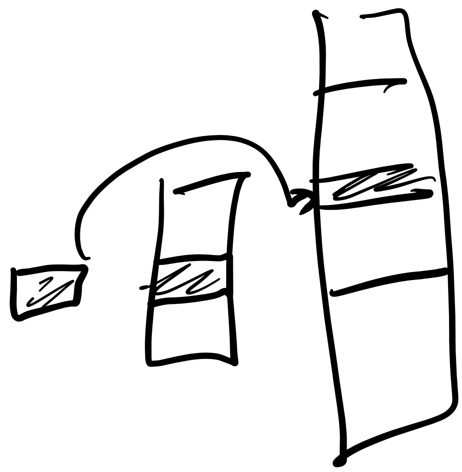
$$\begin{tikzcd} & Z \arrow[dl, two heads] \arrow[dr, hook] \arrow[dd, "\wr"] & \\ X & & Y \\ & Z' \arrow[ul, two heads] \arrow[ur, hook] & \end{tikzcd}$$

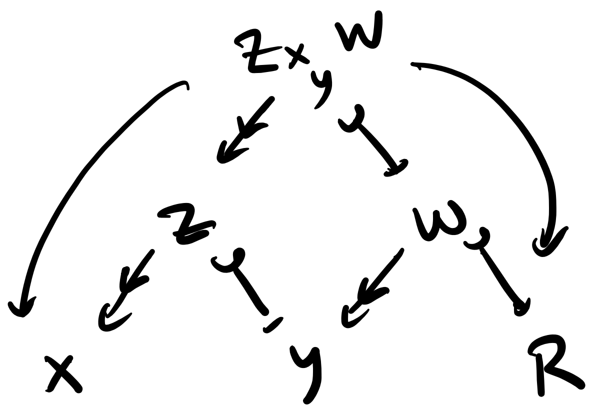


Visual rep. of objects in Ab-cat



Composition of morphisms





Nerve construction

$\mathcal{D}$ category (e.g. $\mathcal{D} = \mathcal{QC}$)

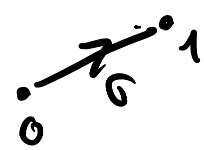
$N\mathcal{D}$ simplicial set.

Simplicial set S has

S_0 set of "vertices" 0-simplices

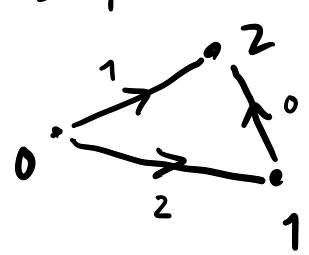
$\partial_1 = s \uparrow \; \uparrow t = \partial_0$

S_1 set of (oriented) edges between 0-simplices



$\uparrow\uparrow\uparrow$

S_2 set of 2simplices (Δ's)



$(N\mathcal{D})_0 = ob(\mathcal{D})$ $\quad a \xrightarrow{f} b$

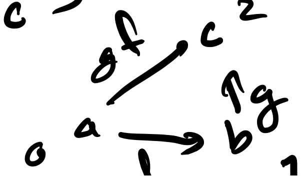
$(N\mathcal{D})_1 = mor(\mathcal{D})$ $\quad \partial_0 f = b$

$\partial_1 f = a$

$(N\mathcal{D})_2 = \{ \text{diagrams } a \xrightarrow{f} b \xrightarrow{g} c \}$

∂_0

(picture as:



$$\begin{array}{c} c \\ \uparrow g \\ b \end{array} \qquad \xrightarrow{\partial_1} \; a \xrightarrow{gf} c \qquad \xrightarrow{\partial_2} \; a \xrightarrow{f} b$$

$$(ND)_n = \{ a_0 \to a_1 \to \cdots \to a_n \}$$

$$\xrightarrow{\partial_0} \; a_1 \to \cdots \to a_n$$

$$\xrightarrow{\partial_i} \; a_0 \to \cdots \to a_{i-1} \to a_{i+1} \to \cdots \to a_n \qquad (a_{i-1} \to a_i \to a_{i+1})$$

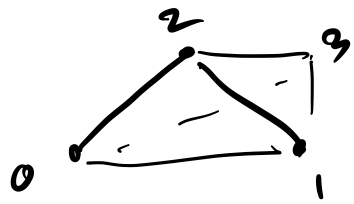
$$\xrightarrow{\partial_n} \; a_0 \to \cdots \to a_{n-1}$$

<u>Brief simplicial set summary</u> (May simplicial objects in alg top)

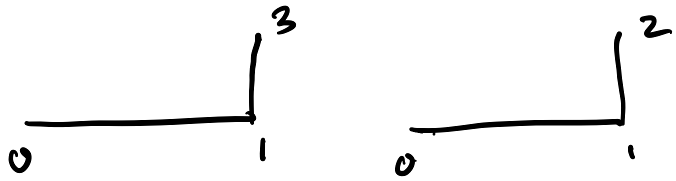
<u>Def</u> A simplicial set S is a collection of sets $S_0, S_1, \ldots$ together with maps $\partial_i : S_n \to S_{n-1}$, $i = 0, \ldots, n$ (faces)

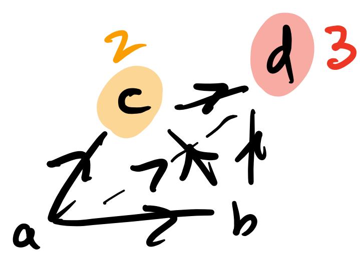
and maps $s_i : S_{n-1} \to S_n$, $i = 0, \ldots, n-1$ satisfying some identities.

$v \qquad v \to v$

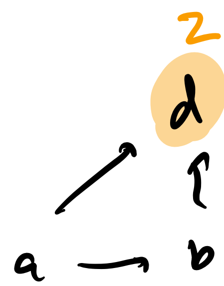


$\sigma \quad \partial_2 \partial_2 \sigma = \partial_2 \partial_3 \sigma$





$\partial_2 \sigma$



Alternately:

Define simplex category

Δ = objects are finite totally ordered sets

morphisms = order preserving ($\leq$)

$\{1,2,3\} \to \{1,2\}$

$1 \to 1$
$2 \to 2$
$3 \to 2$

crush to 2 map

or

$1 \to 2$
$2 \to 2$
$3 \to 2$

$\{1,2\} \to$

$1 \to 1$
$2 \to 3$

← skip 2

$1 \to 2$
$2 \to 3$

← skip 1

or

$1 \to 1$
$2 \to 1$
$3 \to 2$

crush to 1 map

Def A simplicial object in a cat $\mathcal{D}$ is a contravariant functor from Δ to $\mathcal{D}$

$$s\mathcal{D} = \mathrm{Fun}(\Delta^{op}, \mathcal{D})$$

for practical purposes, let $[n] = \{0, \dots, n\}$ w/ standard ordering

funct $\sigma : \Delta^{op} \longrightarrow \mathcal{D}$

is determined by its action on $[n]$

$\sigma([n])$ and

its action on standard maps

$$[n] \xrightarrow[\text{crush } i]{\check{s}_i} [n-1] \qquad \sigma(\check{s}_i) = s_i$$

$$[n-1] \xrightarrow[\text{skip } i]{\check{\partial}_i} [n] \qquad \sigma(\check{\partial}_i) = \partial_i$$

$$\mathrm{Sset} = \mathrm{Fun}(\Delta^{op}, \mathrm{Sets})$$

$S \in$ $\qquad S([n]) = S_n \qquad S_n \xrightarrow{\partial_i} S_{n-1}$

$$S_n \xleftarrow[s_i]{} S_{n-1}$$

Skip to a punchline

$\mathbb{D}$ category $\rightsquigarrow$ $|N\mathbb{D}| = B\mathbb{D}$
top space

<u>Recall:</u> G a gp (discrete) BG - top space w/ $\pi_1 BG = G$

and whose univ cov $\begin{array}{c} EG \\ \downarrow G \\ BG \end{array}$

EG is contractible.

Category of cov. spaces of BG
is eq. to cat of G sets.

Cov spaces $\rightsquigarrow$ G sets

$$\left[\begin{array}{c} Y \leftarrow EG \\ \downarrow \\ BG \end{array}\right] \rightsquigarrow Y_*$$

$$\begin{tikzcd}
Y_* \arrow[r] \arrow[d] & Y \arrow[d] & EG \arrow[l] \arrow[d] \\
* \arrow[r] & BG \arrow[r, equal] & BG & * \arrow[l]
\end{tikzcd}$$

$\mathbb{D}$ = cat w/ one object : $[G]$
morphisms G

$|N[G]| = B[G] = BG$

G set = Functs$([G], \text{Sets})$

In general: BD is a top space such that the cat of cov spaces of BD is nat. equiv. to the cat of functors $D \longrightarrow (\mathrm{Set}, \mathrm{bij})$

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] \arrow[l, bend right, dashed] & 0 \end{tikzcd}$$

$$[M] = [M''] + [M']$$

$$[M] = [M'' \oplus M']$$

$$M \twoheadrightarrow M''$$

$$(M \to M'') \xleftrightarrow{\ Q\ } \begin{tikzcd} & M \arrow[dl] \arrow[dr, "id"] & \\ M'' \arrow[rr, dashed, "Q\,\mathrm{map}"'] & & M \end{tikzcd}$$