Classical K-theory

R some ring (not necessarily comm.)
unital, associative

$K_0(R) \quad K_1(R) \quad K_2(R)$

$\shortparallel$

free ab gp gen by iso classes of f.g. projective modules / ses. relation

$[M] - ([M'] + [M''])$
whenever we have
a SES $0 \to M' \to M \to M'' \to 0$

$K_1(R) = $ Abelianization of $GL_\infty(R) = \varinjlim GL_n(R)$

$$GL_n(R) \hookrightarrow GL_{n+1}(R)$$

$$(*) \longmapsto \begin{bmatrix} * & 0 \\ 0 & 1 \end{bmatrix}$$

Remark: if R has a 2-sided ideal I s.t.
R/I is a division ring then $K_1(R) = (R^*)^{ab}$
via the "Dieudonné determinant"

$$GL_n(R) \to (R^*)^{ab}$$

__Turns out:__ that $[GL_\infty(R), GL_\infty(R)]$ equals subgp gen by "elementary matrices"

$e^n_{ij}(\lambda) \in GL_n(R)$ $\quad i \neq j$

$\lambda \in R$ $\qquad e^n_{ij}(\lambda) \equiv I_n + \lambda e^n_{ij}$

$$e^n_{ij} = \begin{array}{cl} \begin{array}{c} j^{th} \\ \downarrow \end{array} & \\ \begin{bmatrix} & 1 \\ 0 & \end{bmatrix} & \leftarrow i^{th} \end{array}$$

def $E_n(R)$ = subgp gen by $e^n_{ij}(\lambda)$'s.

this is normal & $E_\infty(R) = [GL_\infty(R), GL_\infty(R)]$

__Def of $K_2(R)$__

notice in $E_n(R)$, we have the following relations:

$$e^n_{ij}(\lambda)\, e^n_{ij}(\mu) = e^n_{ij}(\lambda + \mu)$$

$$[e^n_{ij}(\lambda), e^n_{k\ell}(\mu)] = \begin{cases} 1 & \text{if } j \neq k,\ i \neq \ell \\ e^n_{i\ell}(\lambda\mu) & \text{if } j = k,\ i \neq \ell \\ e^n_{kj}(-\mu\lambda) & \text{if } j \neq k,\ i = \ell \end{cases}$$

Def $St_n(R)$ = free gp gen by symbols $x^n_{ij}(\lambda)$

modulo relations

$$x^n_{ij}(\lambda)\, x^n_{ij}(\mu) \left(x^n_{ij}(\lambda+\mu)\right)^{-1}$$

$$[x^n_{ij}(\lambda), x^n_{k\ell}(\mu)] \left(\begin{cases} 1 & \text{if } j \neq k,\ i \neq \ell \\ x^n_{i\ell}(\lambda\mu) & \text{if } j = k,\ i \neq \ell \\ x^n_{kj}(-\mu\lambda) & \text{if } j \neq k,\ i = \ell \end{cases} \right)^{-1}$$

$$St_n(R) \hookrightarrow St_{n+1}(R) \cdots$$

$$St_\infty(R) = St(R)$$

Have a natural hom

$$St_\infty(R) \twoheadrightarrow E_\infty(R) \hookrightarrow GL_\infty(R)$$

Def $K_2(R) = \text{kernel}\left(St_\infty(R) \to GL_\infty(R)\right)$
$= Z(E_\infty(R))$

Milnor's Introduction to Algebraic K-thy

Defines "Milnor square"

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & R_1 \arrow[d] \\ R_2 \arrow[r] & R_0 \end{tikzcd}$$

(Alg. gen. of normal crossings where
$\operatorname{Spec} R = \operatorname{Spec} R_1 \sqcup_{\operatorname{Spec} R_0} \operatorname{Spec} R_2$)

<u>ex:</u>

$$\begin{tikzcd} R = \mathbb{C}[x,y]/_{xy} \arrow[r] \arrow[d] & R_1 = \mathbb{C}[x] = R/(y) \arrow[d] \\ R/(x) = R_2 = \mathbb{C}[y] \arrow[r] & R_0 = \mathbb{C} = R/(x,y) \end{tikzcd}$$

In this case, get a exact seq.

$$0 \to K_0(R) \to K_0(R_1) \times K_0(R_2) \to K_0(R_0)$$
$$\to K_1(R) \to K_1(R_1) \times K_1(R_2) \to K_1(R_0)$$
$$\to K_2(R) \to K_2(R_1) \times K_2(R_2) \to K_2(R_0)$$

<u>Concrete K_2:</u>

Pairing: $K_1(R) \times K_1(R) \longrightarrow K_2(R)$

we have for a commuting pair of elements in group $E(R)$,

$\beta, \alpha \in E(R)$, lift to $St(R)$
$\tilde{\alpha}, \tilde{\beta} \in$

$[\tilde{\alpha}, \tilde{\beta}] \longrightarrow 1$ in $E(R)$

$\alpha \star \beta \equiv [\tilde{\alpha}, \tilde{\beta}] \in K_2(R)$

this element doesn't depend on lifts $\tilde{\alpha}, \tilde{\beta}$ of α, β

pairing $K_1(R) \times K_1(R) \longrightarrow K_2(R)$ via.

$A \in GL_n$ $\quad$ $B \in GL_m$

$$\begin{bmatrix} A \otimes I_m & & \\ & A^{-1} \otimes I_m & \\ & & I_{nm} \end{bmatrix} \star \begin{bmatrix} I_n \otimes B & & \\ & I_n \otimes I_m & \\ & & I_n \otimes B^{-1} \end{bmatrix}$$

$\hat{E}(R)$ $\qquad$ $\hat{E}(R)$

$\equiv$

$A \cdot B \in K_2(R)$

(Turns out: $\begin{bmatrix} A & \\ & A^{-1} \end{bmatrix} \in E(R)$)

In particular, $n = m = 1$, can define

$$a \cdot b = \begin{bmatrix} a & & \\ & a^{-1} & \\ & & 1 \end{bmatrix} \star \begin{bmatrix} b & & \\ & 1 & \\ & & b^{-1} \end{bmatrix} \in K_2(R)$$

$\{a\} \in K_1(R)$ class rep. by $a \in R^\times$

$\{a\} \cup \{b\} = \{a, b\}$

<u>Theorem</u> (Matsumoto): If R is a field then $K_2(F)$ gen. by $\{a, b\}$ $a \in R^\times$ modulo only the relations

- $\{a_1 a_2, b\} = \{a_1, b\} + \{a_2, b\}$
- $\{a, b\} + \{b, a\} = 0$ in $K_2(F)$
- $\{a, b\} = 0$ if $a + b = 1$.

<u>Def</u> A symbol on a field F is a map

$$(\, , \,) : F^\times \times F^\times \longrightarrow A \quad \leftarrow \text{Ab gp. s.t.}$$

- $(a_1 a_2, b) = (a_1, b) + (a_2, b)$
- $(a, b) = -(b, a)$
- $(a, b) = 0$ if $a + b = 1$.

<u>ex</u>:

$$\begin{tikzcd} F^\times \times F^\times \arrow[r] \arrow[d] & H^2(F, \mu_\ell^{\otimes 2}) \\ F^\times/(F^\times)^\ell \times F^\times/(F^\times)^\ell \arrow[r, "\simeq"] & H^1(F, \mu_\ell) \times H^1(F, \mu_\ell) \arrow[u] \end{tikzcd}$$

"Gal Cohom gp"

char $F \nmid \ell$.

$\bigoplus H^\bullet(F,\mu_\ell^{\bullet})$ is a graded ring. and

$$H^1(F,\mu_\ell) \cong F^\times/(F^\times)^\ell$$

(Note $(\ ,\ )$ is a symbol means true $K_2(F) \to A$)

$$\begin{tikzcd}
F^\times \times F^\times \arrow[r, "(\ ,\ )"] \arrow[d, "\{\ ,\ \}"'] & A \\
K_2(F) \arrow[ur, "\exists!"] &
\end{tikzcd}$$

$$\{a,b\} \longmapsto (a,b)_\rho$$

$$K_2(F)/\ell \longrightarrow H^2(F,\mu_\ell^{\otimes 2})$$

We'll be making contact with "Brauer group" parametrizes division algebras.

<u>standard construction:</u> $\quad i^2 = -1 = j^2 \quad ij = -ji$

$a,b \in F^\times \quad (a,b)_{-1} =$ gen by i,j

$$i^2 = a \quad j^2 = b \quad ij = -ji$$

more generally if $\rho \in F^\times$ is a prim ℓ^{th} root of 1

$(a,b)_\rho$: gen by $i,j \quad i^\ell = a \quad j^\ell = b \quad ij = \rho j i$

"symbol algebra"

if $\mu_\ell \simeq \mathbb{Z}/\ell$

roots of 1 in F

$$H^2(F, \mu_\ell^{\otimes 2}) = H^2(F, \mu_\ell) = Br(F)[\ell]$$

Another famous symbol

The tame symbol

if F is a discretely valued field $v: F^\times \to \mathbb{Z}$

res field k

$$T_v : F^\times \times F^\times \longrightarrow k^*$$

$$T_v(a,b) = (-1)^{v(a)v(b)} \overline{\left(\frac{a^{v(b)}}{b^{v(a)}}\right)}$$

$$v(a^{v(b)}) = v(b) \cdot v(a)$$

aka "residue" or "ramification" map

If $\operatorname{char} F = p$

Define $\Omega^1_F \equiv \Omega^1_{F/\mathbb{Z}} = \dfrac{\text{free ab. gp gen. by } \{a\,db \mid a,b \in F\}}{\begin{array}{l} a\,dn = 0 \text{ if } n \in \mathbb{Z} \\ a\,d(b_1+b_2) = a\,db_1 + a\,db_2 \\ (a_1+a_2)\,db = \cdots \\ a\,d(bc) = ac\,db + ab\,dc \\ \vdots \end{array}}$

$H^1(F,\mu_p) \hookrightarrow \Omega^1_F$

$da^r = r a^{r-1}\,da$

Def $\Omega^2_F = \Lambda^2_{\mathbb{Z}} \Omega^1_F$

$$F^\times \times F^\times \longrightarrow \Omega^2_F$$

$$(a,b) \longmapsto \frac{da}{a} \wedge \frac{db}{b}$$

In general one can define, (with a hopeful outlook)

$$K^M_n(F) = \frac{\text{free ab gp gen. by "symbols" } \{a_1,\dots,a_n\}}{\text{relations: linear in each variable}}$$

$\{a_1 a_1', a_2, \dots, a_n\}$

$" \{a_1, a_2, \dots, a_n\} + \{a_1', a_2, \dots, a_n\}$

and 0 if two terms add to 1

$\bigoplus K^M_\bullet(F)$ $\qquad$ $K^M_0(F) = \mathbb{Z}$

$K_1^M(F) = F^*$ $\quad$ $K_2^M(F) = K_2(F)$

$K_3^M \dots$? useful def.

it'll turn out, $K_n^M(F) \longrightarrow K_n(F)$

<u>not isos</u>! $\quad$ actual Quillen K-gps

val: $\quad F^* \longrightarrow \mathbb{Z}$

$K_1(F) \longrightarrow K_0(k)$

$F^* \times F^* \longrightarrow k^*$

tame symbol $\quad K_2(F) \longrightarrow K_1(k)$

$$\coprod_{x \in X} K_2^M(x) \longrightarrow \coprod_{x \in X} K_1^M(x) \xrightarrow{\text{div}} \coprod_{x \in X} K_0(x) \to 0$$

$CH(X,1)$ $\qquad$ from here $CH(X)$

"Rost's Chow gps w/ Coefficients"

Elman, Karpinka, Medkyeo Qtm.