Wavelet transforms for unbounded signals

part 1 : frequency spectrum / power spectral density function.

Def $\ell(\mathbb{Z})$ = vector space of discretely sampled signals

vectors written as

$$\ldots, x[-2], x[-1], x[0], x[1], x[2], \ldots$$

Def $\text{support}(x) = \{k \mid x[k] \neq 0\}$

Def $\ell_0(\mathbb{Z})$ = subspace of vectors w/ finite support

Def $\ell(\mathbb{Z}/N\mathbb{Z})$ = subspace of N-periodic signals
($x[k] = x[N+k]$ all k)

We'd like to describe the "frequency spectrum" of an element $x \in \ell_0(\mathbb{Z})$.

Recall:

For a periodic signal x, defined

$$\hat{x}[k] = \sum_{j=0}^{N-1} x[j]\, \rho^{-jk}$$

$\sim$ coeff of E, $E[k] = \rho^{jk}$

Recall:

For aperiodic signal x, defined

$$\hat{x}[k] = \sum_{j=0}^{N-1} x[j]\, e^{-jk} = \sum_{j=1}^{N-1} x[j] \left(e^{-\frac{2\pi i}{N}}\right)^{jk}$$

$$= \sum_{j=1}^{N-1} x[j] \left(e^{-2\pi i k/N}\right)^{j}$$

Let $\nu = \dfrac{2\pi k}{N}$

Recall:

For a periodic signal x, defined

$$\hat{x}[k] = \sum_{j=0}^{N-1} x[j] \, \rho^{-jk} = \sum_{j=0}^{N-1} x[j] \, e^{-i\nu jk}$$

$\sim$ coeff of E_k, $E_k[j] = \rho^{jk} = e^{i\nu jk}$

Setting $\omega = \nu k$, see amount of $j \longrightarrow e^{i\omega j}$

is $\sim \sum_{j=0}^{N-1} x[j] \, e^{-i\omega j}$

$\rightsquigarrow$ Amount of frequency ω $(j \mapsto e^{i\omega j})$

proportional to $\sum_{j \in \mathbb{Z}} x[j]\, e^{-i\omega j}$ (morally)

Def For a signal $x \in l_0(\mathbb{Z})$, the z-transform of x

is $$X(z) = \sum_{j \in \mathbb{Z}} x[j]\, z^{-j}$$

amount of freq. $\omega \sim X(e^{-i\omega})$

Def $P(\omega) = |X(e^{i\omega})|^2$

is the power spectral density function of x

~ $P(\omega)$ relates to how much ω-like stuff is in x ...