2-Dimensional Wavelet Transforms

$\boxed{\text{1-d signals}}$

typical example: sounds

$z[k]$ = signal at time k

$\boxed{\text{2-d signals}}$

typical example: images (grayscale)

$z[j,k]$ = signal at point with coordinates (j,k)

1-d signals

represented as vectors $z[k] \in \mathbb{C}^N$

$$\begin{bmatrix} z[0] \\ \vdots \\ z[N-1] \end{bmatrix}$$

2-d signals

represented as matrices $z[j,k] \in M_N(\mathbb{C})$

$$\begin{bmatrix} z[0,0] & z[0,1] & \cdots & z[0,N-1] \\ \vdots & & & \vdots \\ z[N-1,0] & \cdots & \cdots & z[N-1,N-1] \end{bmatrix}$$

$\cong \mathbb{C}^{N^2}$

Plot: Given a 1-dimensional wavelet transform, how can we use it to obtain a 2-dim'l transform?

$$T_a x = \begin{bmatrix} s \\ d \end{bmatrix} \qquad T_a : \mathbb{C}^N \longrightarrow \mathbb{C}^N$$

$$N = 2m \qquad T_a \in M_N(\mathbb{C})$$

$$\rightsquigarrow$$

$$T_a^{2\text{-}d} : \mathbb{C}^{N^2} \longrightarrow \mathbb{C}^{N^2}$$

Idea: We apply the wavelet transform

- vertically (to each column)
- horizontally (to each row)

$$\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = Z \rightsquigarrow \left[\begin{array}{c} v.tr(Z) \\ \hline v.det(Z) \end{array}\right] \rightsquigarrow \begin{bmatrix} htr(vtr(Z)) & hdet(vtr(Z)) \\ htr(vdet(Z)) & hdet(vdet(Z)) \end{bmatrix}$$

$$
\begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix}
$$

$$
\|
$$

$$
\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = z \rightsquigarrow \begin{bmatrix} v.tr(z) \\ \hline v.det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} htr(vtr(z)) & hdet(vtr(z)) \\ htr(vdet(z)) & hdet(vdet(z)) \end{bmatrix}
$$

trend

$$\begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix} = \begin{bmatrix} h.tr(v.tr(z)) & h.det(v.tr(z)) \\ h.tr(v.det(z)) & h.det(v.det(z)) \end{bmatrix}$$

$$\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = z \rightsquigarrow \begin{bmatrix} v.tr(z) \\ \hline v.det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} h.tr(v.tr(z)) & h.det(v.tr(z)) \\ h.tr(v.det(z)) & h.det(v.det(z)) \end{bmatrix}$$

$$\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = z \rightsquigarrow \begin{bmatrix} v.tr(z) \\ \hline v.det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} htr(vtr(z)) & hdet(vtr(z)) \\ htr(vdet(z)) & hdet(vdet(z)) \end{bmatrix}$$

$$= \begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix}$$

hor. trend
v. details $\longrightarrow Z_{ds}$

see changes under v. movement

hor. trend
v. details

$$\begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix}$$

$$\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = z \rightsquigarrow \begin{bmatrix} v.tr(z) \\ \hline v.det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} htr(vtr(z)) & h.det(vtr(z)) \\ htr(vdet(z)) & h.det(vdet(z)) \end{bmatrix}$$

$\boxed{\text{horizontal features}}$ = see changes under v. movement

hor. trend
v. details

$$\begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix}$$

$$\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = z \rightsquigarrow \begin{bmatrix} v.tr(z) \\ \hline v.det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} htr(vtr(z)) & hdet(vtr(z)) \\ htr(vdet(z)) & hdet(vdet(z)) \end{bmatrix}$$

vertical trend
horizontal detail
= vertical features

$$\begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = Z \rightsquigarrow \begin{bmatrix} v.tr(Z) \\ \hline v.det(Z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} htr(vtr(Z)) & h.det(vtr(Z)) \\ htr(vdet(Z)) & h.det(vdet(Z)) \end{bmatrix}$$

detects changes when **both** vert. & horiz. move

$$
\begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix}
$$

$$
\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = z \rightsquigarrow \begin{bmatrix} v.tr(z) \\ \hline v.det(z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} htr(vtr(z)) & hdet(vtr(z)) \\ htr(vdet(z)) & hdet(vdet(z)) \end{bmatrix} = \begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix}
$$

diagonal features

$\parallel$

detects changes when **both** vert. & horiz. move

$$\begin{bmatrix} Z_{ss} & Z_{sd} \\ Z_{ds} & Z_{dd} \end{bmatrix}$$

$\parallel$

$$\begin{bmatrix} z[0,0] & \cdots & z[0,N-1] \\ \vdots & & \vdots \\ z[N-1] & \cdots & z[N-1,N-1] \end{bmatrix} = Z \rightsquigarrow \begin{bmatrix} v.tr(Z) \\ \hline v.det(Z) \end{bmatrix} \rightsquigarrow \begin{bmatrix} h.tr(v.tr(Z)) & h.det(v.tr(Z)) \\ h.tr(v.det(Z)) & h.det(v.det(Z)) \end{bmatrix}$$

Q: How to express T_a^{2d} in terms of matrix operators ?

Linear Algebra

$$T \cdot \begin{bmatrix} | & | & & | \\ v_0 & v_1 & \cdots & v_{N-1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Tv_0 & Tv_1 & \cdots & Tv_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} - & w_0 & - \\ - & w_1 & - \\ & \vdots & \\ - & w_{N-1} & - \end{bmatrix} T^t = \left(T \begin{bmatrix} | & & | \\ w_0 & \cdots & w_{N-1} \\ | & & | \end{bmatrix} \right)^t = \left(\begin{bmatrix} | & & | \\ Tw_0 & \cdots & Tw_{N-1} \\ | & & | \end{bmatrix} \right)^t$$

Linear Algebra

$$T \cdot \begin{bmatrix} | & | & & | \\ v_0 & v_1 & \cdots & v_{N-1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Tv_0 & Tv_1 & \cdots & Tv_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} - & w_0 & - \\ - & w_1 & - \\ & \vdots & \\ - & w_{N-1} & - \end{bmatrix} T^t = \left(T \begin{bmatrix} | & & | \\ w_0 & \cdots & w_{N-1} \\ | & & | \end{bmatrix} \right)^t = \begin{bmatrix} - & Tw_0 & - \\ - & \vdots & - \\ - & Tw_{N-1} & - \end{bmatrix}$$

$z \rightsquigarrow T_a z$ = vertical/column wavelet transform

$z \rightsquigarrow z(T_a)^t$ = horizontal/row wavelet transform

$z \rightsquigarrow T_a z (T_a)^t$ = 2-D wavelet transform

$$T_a z (T_a)^t = \left[\begin{array}{c|c} z_{ss} & z_{sd} \\ \hline z_{ds} & z_{dd} \end{array}\right]$$