Wavelet transforms for unbounded signals

part 2: towards wavelets from filters

Applying our wavelet transforms to unbounded signals

$$\ell(\mathbb{Z}/N) \rightsquigarrow \ell_0(\mathbb{Z})$$

$$\boxed{\ell(\mathbb{Z}/N\mathbb{Z}) \text{ case}}$$

Haar $\quad x \to \begin{bmatrix} x_{even} \\ x_{odd} \end{bmatrix} \rightsquigarrow \frac{1}{2}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} x_{even} \\ x_{odd} \end{bmatrix} = \begin{bmatrix} s \\ d \end{bmatrix}$

CDF(2,2) $\quad x \mapsto \begin{bmatrix} x_{even} \\ x_{odd} \end{bmatrix} \rightsquigarrow \mathrm{DUP}\begin{bmatrix} x_{even} \\ x_{odd} \end{bmatrix} = \begin{bmatrix} s \\ d \end{bmatrix}$

$$\frac{1}{4\sqrt{2}}\begin{bmatrix} -S^{-1}+6I-S & 2I+2S \\ -2S^{-1}-2I & 4I \end{bmatrix}$$

$$\boxed{\ell_0(\mathbb{Z}) \text{ case}}$$

<u>Def</u> $x_{2\downarrow}[k] = x[2k]$

and $\boxed{2\downarrow} : \ell_0(\mathbb{Z}) \longrightarrow \ell_0(\mathbb{Z})$

$x \longmapsto x_{2\downarrow}$

<u>Def</u> $(Sx)[k] = x[k-1]$

$x_0 = \boxed{2\downarrow}\, x$

"x_{even}"

$x_1 = \boxed{2\downarrow}\, S^{-1} x$

"x_{odd}"

$$x_0[k] = \left(\boxed{2\downarrow}\, x\right)[k] = x[2k]$$

$$x_1[k] = \left(\boxed{2\downarrow}\, S^{-1}x\right)[k] = \left(S^{-1}x\right)[2k]$$

$$= x[2k+1]$$

Use these like x_{even} & x_{odd}.

Standard Wavelet Transforms on $\ell_0(\mathbb{Z})$

Haar: $x \rightsquigarrow \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \rightsquigarrow \frac{1}{2}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} s \\ d \end{bmatrix}$

So, $s = \frac{1}{2}(x_0 + x_1)$ $\quad d = \frac{1}{2}(x_1 - x_0)$

$$s[k] = \frac{1}{2}\left(x_0[k] + x_1[k]\right) = \frac{1}{2}\left(x[2k] + x[2k+1]\right)$$

$$d[k] = \frac{1}{2}\left(x[2k+1] - x[2k]\right)$$

Standard Wavelet Transforms on $l_0(\mathbb{Z})$

$$\mathrm{CDF}(2,2): x \rightsquigarrow \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \rightsquigarrow \frac{1}{4\sqrt{2}} \begin{bmatrix} -S^{-1}+6I-S & 2I+2S \\ -2S^{-1}-2I & 4I \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

$$= \begin{bmatrix} s \\ a \end{bmatrix}$$

$$\rightsquigarrow s[k] = -S^{-1}x_0[k] + 6x_0[k] - Sx_0[k] + 2x_1[k] + 2Sx_1[k]$$

$$= -x[2k+2] + 6x[2k] - x[2k-2] + 2x[2k+1] + 2x[2k-1]$$

$\ldots$

But why?

Alternate construction of wavelets:

$$\begin{tikzcd}
 & H_0 x \arrow[r] & \boxed{2\downarrow} H_0 x = s \\
x \arrow[ur] \arrow[dr] & & \\
 & H_1 x \arrow[r] & \boxed{2\downarrow} H_1 x = d
\end{tikzcd}$$

H_0 = lowpass filter

H_1 = highpass filter

Filters (FIR = finite impulse response)

Described by convolutions

$$x \longmapsto h * x$$

<u>Def</u> $h * x = \sum_n \left(\sum_{j+k=n} h[j]\, x[k] \right) e_n$

<u>Alternately</u> <u>Def</u> $e_j * e_k = e_{j+k}$ & foil (distribute)

$$(e_{-1} + 3e_2 + e_4) * (e_1 - e_3)$$

$$= \underbrace{e_{-1} * e_1}_{e_0} - \underbrace{e_{-1} * e_3}_{e_2} + 3\underbrace{e_2 * e_1}_{e_3} - 3\overbrace{e_2 * e_3}^{e_5} + \overbrace{e_4 * e_1}^{e_5} - \underbrace{e_4 * e_3}_{e_7}$$

$$= e_0 - e_2 + 3e_3 - 3e_5 + e_5 - e_7$$

$$= e_0 - e_2 + 3e_3 - 2e_5 - e_7$$

ex i

$$h_0 = e_{-1} + e_0 \qquad h_0 * x = x + Sx$$

$$h_1 = e_{-1} - e_0 \qquad h_1 * x = -x + Sx$$

$$s = \boxed{2\downarrow}(h_0 * x)[k] = x[2k] + x[2k+1]$$

$$d = \boxed{2\downarrow}(h_1 * x)[k] = -x[2k] + x[2k+1]$$

If we have a filter $\ell_0(\mathbb{Z}) \to \ell_0(\mathbb{Z})$, given by h

$$X(z)\,H(z) = z\text{-transform of } x * h \;!$$

$\Rightarrow$ Frequencies are affected by h in proportion to their presence in h.

Haar:

$$h_0 = e_{-1} + e_0 \rightsquigarrow H_0(z) = z^{-1} + 1$$

$$h_1 = e_{-1} - e_0 \rightsquigarrow H_1(z) = z^{-1} - 1$$

Power spectral densities:

$$P_0(\omega) = |H_0(e^{i\omega})|^2 = (e^{-i\omega} + 1)(e^{i\omega} + 1) = 2 + 2\cos\omega$$

$$P_1(\omega) = 2 - 2\cos\omega$$

$\omega = 0$ low

$\omega = \pi$ high

Haar inconvenience

H_0 cuts off highest freq. well

cuts off high-ish freq so-so

H_1 ... similar.

Quest: Find filters which do this better & which are **reversible!**