Graduate Algebra, Midterm Exam Practice Sheet

1. Give an example of a ring R and an ideal $I \triangleleft R$ such that I is prime, but not maximal, and not principal.

2. Give an example of a group G ~~with~~ **and** a normal subgroup N, such that G cannot be written as a semidirect product of H and N for any subgroup $H < G$.

3. Let F be a field, and consider the rings $F[x]$ of polynomials and $F[[x]]$ of power series. Let $S \subset F[x]$ be the multiplicative set consisting of those polynomials with a nonzero constant term.
   (a) Show that the inclusion $F[x] \to F[[x]]$ extends to a homomorphism $F[x][S^{-1}] \to F[[x]]$.

   (b) Show that $F[[x]]$ is a PID.

   Hint: Show that if $f(x) = a_d x^d + a_{d+1} x^{d+1} + \cdots$ with $a_d \neq 0$, and if $g(x) = b_e x^e + b_{e+1} x^{e+1} + \cdots$ with $b_e \neq 0$ and $d \leq e$, then $f(x) | g(x)$.

4. Let G be a group of order 30, and suppose G has exactly 15 Sylow subgroups of order 2. Show that G must be isomorphic to a dihedral group $\langle \sigma, \tau \mid \sigma^{15} = 1, \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$.

5. Let $X \subset \mathbb{R}$ be a collection of disjoint closed intervals, and let R be the ring of continuous functions from X to $\mathbb{R}$. Recall that an element $e \in R$ is called idempotent if $e^2 = e$. Show that the only idempotents in R are 0 and 1 if and only if X is connected.

6. Let R be a commutative ring. Recall that an element $x \in R$ is nilpotent if $x^n = 0$ for some $n > 0$. Show that the set of nilpotent element of R form an ideal of R.

7. Show that if G is a group with $G/Z(G)$ cyclic, then G is Abelian.

8. Show that every group of order 45 is Abelian.

9. Show that in any group, G and prime number p dividing the order of G, if P is a Sylow-p subgroup, then the normalizer $N_G(P)$ of P in G cannot be a normal subgroup unless $N_G(P) = G$.