Graduate Algebra, Fall 2019, Final Exam

Instructor: Daniel Krashen

Name ____

By signing below, I pledge that the work on this exam is my own,
and was done without outside help.

Signature ____

Please hand only only clearly written work, not scratch paper.

Please refrain from using notes or the textbook. This will ensure that your results give you an accurate portrayal of your readiness for the qualifying exam.

You may hand in your work either on separate pieces of paper, or written on a printed version of this exam, or TeXed up separately. You may scan or photograph your completed exam and email it to me, or simply hand in a physical copy to my office or departmental mailbox. If you can think of another way you would rather hand it in, let me know.

Due 11:59pm, Thursday December 12, 2019.

This exam has 7 questions, for a total of 70 points.

1. (10 points) (a) Show that the groups $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ are not isomorphic.

(b) Show that the groups $\mathbb{Q}$ and $\mathbb{Q} \oplus \mathbb{Q}$ are not isomorphic.

(c) Show that the groups $\mathbb{Q}$ and $\mathbb{Q}/Z$ are not isomorphic.

2. (10 points) Suppose that T is a complex $n \times n$ matrix such that $T^n = 1$ for some $n > 0$. Show that T is diagonalizable.

   remember – we are not using any of the facts from the representation theory portion of the course!

3. (10 points) Let G be a group, and suppose that K and H are subgroups such that $K \subset H$ and $H \subset N_G(K)$. Show that H is also in the normalizer of $C_G(K)$.

4. (10 points) Let T be an $n \times n$ matrix over the field $\mathbb{F}_2$ with two elements. Suppose that $T^2 = 1$. Describe the possible Jordan forms of T.

5. (10 points) Show that no group of order 56 can be simple.

6. (10 points) Show that a group of order pqr for distinct primes p, q, r must have a non-trivial normal Sylow subgroup.

7. (10 points) Give an example of a matrix over the field of rational numbers, which cannot be put into Jordan canonical form.