Goal:

Theorem: If $f: X \to Y$ is a continuous map between metric spaces and $S \subset X$ is compact, then $f(S)$ is compact.

Recall: If $S \subset X$ is compact then S is closed & bounded.

Theorem implies that the image of a compact set is bounded & contains all its limit points.

Ended w/:

Lemma: $f: X \to Y$ continuous iff $\forall U \subset Y$ open, $f^{-1}(U)$ is open in X.

Proof of the theorem:

Suppose $f: X \to Y$ continuous, $S \subset X$ compact.

WTS: $f(S)$ is compact.

Want to show: whenever we have a cover of $f(S)$ by open sets U_i, $i \in I$, then $\exists$ finite subcollection of open sets which cover.

Consider $f^{-1}(U_i)$. by def of continuity $(*)$ these are open in X & notice they cover S!

Since if $x \in S$ then $f(x) \in f(S)$

$f(x) \in U_i$ some i

$\Rightarrow x \in f^{-1}(U_i)$

therefore since S is compact $\exists J \subset I$

J finite, s.t. $f^{-1}(U_i)$ $i \in J$, these cover S.

But now, claim U_i, $i \in J$ cover $f(S)$

since if $y = f(x) \in f(S)$ for $x \in S$

$x \in f^{-1}(U_i)$, $i \in J$

since these cover S

$\Rightarrow f(x) \in U_i$ so $y \in U_i$

$\square$

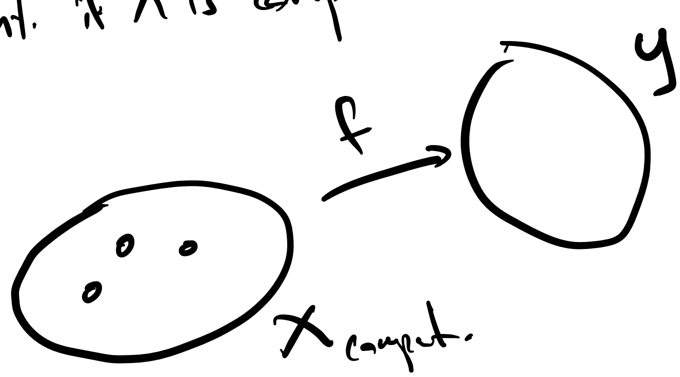
Uniform continuity

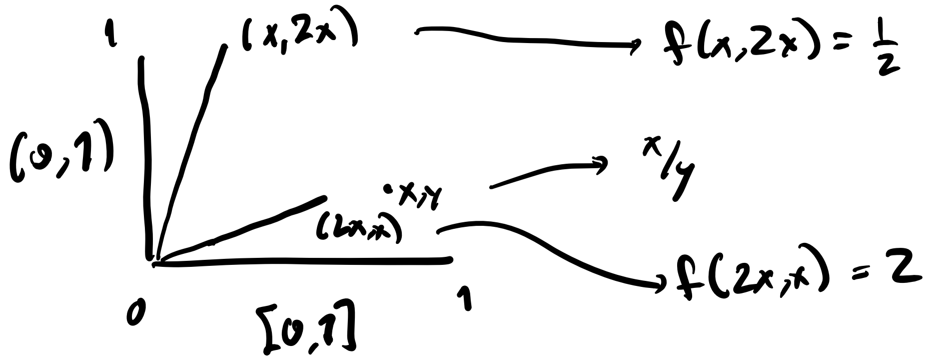
Def.: $f: X \longrightarrow Y$ uniformly cont. if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. whenever $d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \varepsilon$.

Main important fact: (Theorem)

cont = unif. cont. if X is compact.

Pf:





Choose $\varepsilon = 1$. Claim, $\forall \delta > 0 \ \exists p, q \in [0,1] \times (0,1)$

s.t. $d(p,q) < \delta$ but $d(f(p), f(q)) \geq 1$

want to choose x s.t. $p = (x, 2x)$, $q = (2x, x)$ then $d(p,q) < \delta$

$$d(f(p), f(q)) = |2 - \tfrac{1}{2}| = 1/2 \geq 1$$

$$d(p,q) = \sqrt{(x-2x)^2 + (2x-x)^2} = \sqrt{x^2 + x^2} = \sqrt{2}|x| = \sqrt{2}x$$

choose $x < \frac{\delta}{\sqrt{2}}$ $\qquad \underset{d(p,q)}{\underset{\parallel}{\sqrt{2}x}} < \delta$

WTS $\exists \varepsilon > 0$ s.t. $\forall \delta > 0 \ \exists p, q \in [0,1] \times (0,1)$

s.t. $d(p,q) < \delta$ but $d(f(p), f(q)) \geq \varepsilon$

Why? pick $\varepsilon = 1$ $\quad p = \left(2\frac{\delta}{2\sqrt{2}}, \frac{\delta}{2\sqrt{2}}\right)$ $\quad q = \left(\frac{\delta}{2\sqrt{2}}, \frac{2\delta}{2\sqrt{2}}\right)$

$$d(p,q) = \sqrt{\left(\frac{2\delta}{2\sqrt{2}} - \frac{\delta}{2\sqrt{2}}\right)^2 + \left(\frac{\delta}{2\sqrt{2}} - \frac{2\delta}{2\sqrt{2}}\right)^2}$$

$$= \sqrt{\left(\frac{\delta}{2\sqrt{2}}\right)^2 + \left(\frac{\delta}{2\sqrt{2}}\right)^2} = \sqrt{\frac{\delta^2}{4}} = \frac{1}{2}\delta$$

and $d(f(p), f(q)) = \left| \left(\frac{2\delta}{2\sqrt{2}}\right) \Big/ \left(\frac{\delta}{2\sqrt{2}}\right) - \left(\frac{\delta}{2\sqrt{2}}\right) \Big/ \left(\frac{2\delta}{2\sqrt{2}}\right) \right|$

$$= \left|2 - \frac{1}{2}\right| = 1\frac{1}{2} \geq 1.$$

<u>Theorem:</u> If $f: X \to Y$ continuous, X compact then f is uniformly continuous.

<u>Pf:</u> Choose $\varepsilon > 0$. WTS $\exists\, \delta > 0$ s.t. if $d(x_1, x_2) < \delta$ then $d(f(x_1), f(x_2)) < \varepsilon$.

$\forall\, x \in X$. $\exists\, \delta_x$ s.t. $d(x', x) < \delta_x$ then $d(f(x'), f(x)) < \frac{\varepsilon}{2}$.

Consider $U_x = B_{\delta_x}(x)$

U_x, $x \in X$ cover X. But X is compact.

By Lebesgue cov lemma $\exists\, \delta$ s.t. $\forall x_0$ $B_\delta(x_0)$ is in one of the U_x $x \in X$.

Claim: if $d(x_1,x_2) < \frac{\delta}{2}$ then $d(f(x_1),f(x_2)) < \varepsilon$

Pf of claim:

if $d(x_1,x_2) < \frac{\delta}{2}$ then know $B_\delta(x_1) \subset U_x$ some $x \in X$

$$U_x = B_{\delta_x}(x)$$

$$x_2 \in B_{\frac{\delta}{2}}(x_1) \subset B_\delta(x_1)$$

Now since $x_i \in U_x = B_{\delta_x}(x)$, $i=1,2$, we know $d(f(x_i), f(x)) < \frac{\varepsilon}{2}$

so
$$d(f(x_1), f(x_2)) \le d(f(x_1), f(x)) + d(f(x_2), f(x))$$
$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = 2\frac{\varepsilon}{2} = \varepsilon \quad \square$$

<u>Example result:</u>

Suppose X is compact $x \in X$.

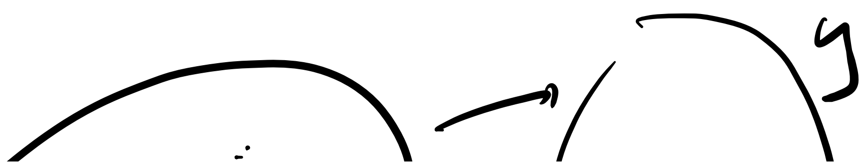
$f: X \setminus \{x\} \longrightarrow Y$ continuous.

then $\exists\, \tilde{f}: X \longrightarrow Y$ continuous s.t. $\tilde{f}(x') = f(x')$ all $x' \in X \setminus \{x\}$

if and only if

f is uniformly continuous.

Pf:



Choose a seq (x_i) s.t.

$\lim x_i = x$

define $\tilde{f}(x) = \lim_{i\to\infty} f(x_i)$