f continuous at x_0 if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

if $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

f uniformly continuous if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $f: [a,b] \to \mathbb{R}$

Suppose X, Y metric spaces

<u>Def.</u> we say a function $f: X \to Y$ is continuous at $x_0 \in X$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. whenever $d(x, x_0) < \delta$ we have $d(f(x), f(x_0)) < \varepsilon$.

we say f is cont. if it is continuous at every point $x_0 \in X$.

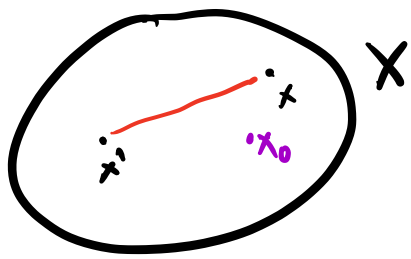
<u>ex:</u> X a metric space, $x' \in X$ define

$f: X \to \mathbb{R}$ via $f(x) = d(x, x')$

why is this continuous?

wts given $x_0 \in X$, $\varepsilon > 0$, set $\delta = \varepsilon$

if $x \in X$ w/ $d(x, x_0) < \varepsilon$ $\quad d($



$f(x) = d(x', x)$

Proposition a function $f: X \to Y$ between metric spaces is continuous if and only if it "preserves limits"

i.e. whenever (a_n) a seq in X converging to a then we have $(f(a_n))$ converge to $f(a)$

i.e. $f(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} f(a_n)$

Proof: suppose f is cont, why does it preserve limits?

suppose $\lim_{n\to\infty} a_n = a$, WTS $\lim_{n\to\infty} f(a_n) = f(a)$

i.e. WTS $\forall \varepsilon > 0 \; \exists N > 0$ s.t. $n \geq N \Rightarrow$
$d(f(a_n), f(a)) < \varepsilon$

since f is cont at a $\exists \delta$ s.t. $d(b,a) < \delta$
then $d(f(b), f(a)) < \varepsilon$

since $\lim_{n\to\infty} a_n = a$ $\exists N$ s.t. $n \geq N$ then

$$d(a_n, a) < \delta$$

but now, for $n \geq N$, $d(a_n,a) < \delta \implies d(f(a_n), f(a)) < \varepsilon$.

Conversely, suppose that f preserves limits.
why is it continuous?

Suppose it wasn't, say $\exists x_0 \in X$ s.t. $\exists \varepsilon > 0$
s.t. $\forall \delta > 0$ $\exists x$, $d(x, x_0) < \delta$ but $d(f(x), f(x_0)) \geq \varepsilon$.

in particular, for $\delta = \frac{1}{n}$ $\exists x_n$ s.t. $d(x, x_n) < \frac{1}{n}$
but $d(f(x), f(x_0)) \geq \varepsilon$

but notice $\lim_{n\to\infty} x_n = x$ but $\lim_{n\to\infty} f(x_n) \neq f(x)$

since $d(f(x_n), f(x)) \geq \varepsilon$!

contradicts f preserving limits! $\lightning$

$$(a_1, a_2, \ldots, a_n) \longleftrightarrow 2^{a_1} 3^{a_2} \cdots p_n^{a_n}$$

Theorem: Suppose $f: X \to Y$ continuous map between metric spaces, an $S \subset X$ is compact. Then $f(S)$ is also compact!

Cor: if $S \subset X$ compact, choose $x_0 \in X$ arb. and consider $f: X \to \mathbb{R}$

$x \mapsto d(x, x_0)$

$f(S) \subset \mathbb{R}$ compact. ($\Rightarrow f(S)$ closed & bounded)

$\Rightarrow d(x, x_0)$ has an upper bnd $\Rightarrow S$ bounded.

Cor if $S \subset X$ compact, any cont real valued fcn $f: X \to \mathbb{R}$ attains a max value!

Step 1: Reformulation of continuity

Lemma: $f: X \to Y$ map between metric spaces is continuous iff $\forall U \subset Y$ open, $f^{-1}(U)$ is open.

$f^{-1}(U) = \{x \in X \mid f(x) \in U\}$

Pf: temp. call above condition "continuous☆"

Suppose f is cont☆. WTS f is cont.

for $x_0 \in X$, choose $\varepsilon > 0$ WTS $\exists \delta$ s....

consider $B_\varepsilon(f(x_0))$. this is open.

$\Rightarrow f^{-1}(B_\varepsilon(f(x_0)))$ is open in X

and notice $x_0 \in f^{-1}(B_\varepsilon(f(x_0)))$

(i.e. $f(x_0) \in B_\varepsilon(f(x_0))$)

but since $f^{-1}(B_\varepsilon(f(x_0)))$ is open, x_0 is an interior pt.

$\Rightarrow \exists \delta > 0$ s.t. $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$

but therefore if $d(x, x_0) < \delta \Rightarrow x \in B_\delta(x_0)$

$\Rightarrow f(x) \in B_\varepsilon(f(x_0)) \Rightarrow d(f(x), f(x_0)) < \varepsilon$

D.

Suppose f is cont why is f cont A?

to show f is cont A. choose $U \subset Y$ open

WTS $f^{-1}(U)$ is open in X.

i.e. if $x \in f^{-1}(U)$, WTS x is an interior pt.

i.e. $\exists \delta > 0$ s.t. $B_\delta(x) \subset f^{-1}(U)$.

but since U is open, $f(x) \in U$ $f(x)$ is an int. pt

of U. $\Rightarrow \exists \varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \subset U$

By continuity at x $\exists \delta > 0$ s.t. $d(x, x') < \delta$

$\Rightarrow d(f(x), f(x')) < \varepsilon$

But now claim $B_\delta(x) \subset f^{-1}(U)$!

Since if $x' \in B_\delta(x) \Rightarrow d(x,x') < \delta$

$\Rightarrow d(f(x), f(x')) < \varepsilon \Rightarrow f(x') \in B_\varepsilon(f(x))$

$\cap$

U

so $f(x') \in U \Leftrightarrow x' \in f^{-1}(U)$.

so $B_\delta(x) \subset f^{-1}(U)$ so x an int pt.

$\Rightarrow f^{-1}(U)$ open $\square$

square of victory

Theorem: Suppose $f: X \to Y$ continuous map between metric spaces, and $S \subset X$ is compact. Then $f(S)$ is also compact!