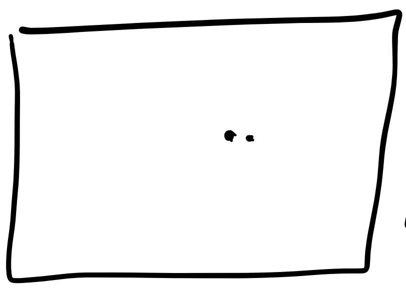
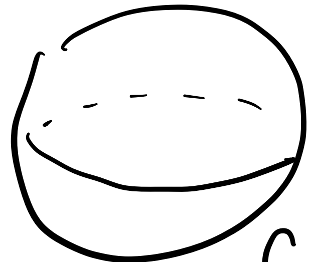
dim 1

$\mathbb{R}$



$\mathbb{R}^2$

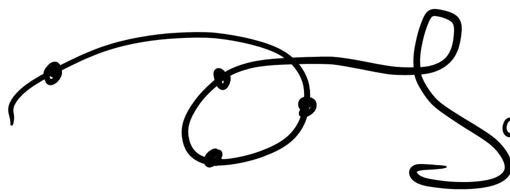
dim 2



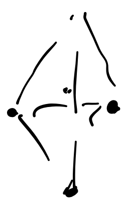
dim 2

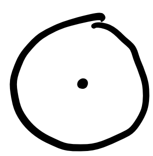
C

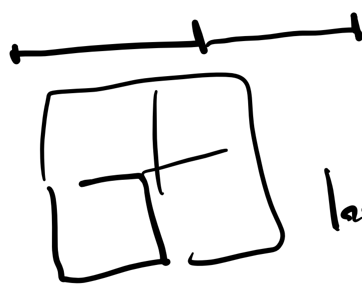
dim 3



dim 1 ?

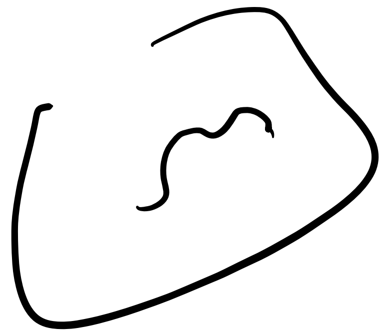






$\log_n \left(\text{\# Balls rad } \varepsilon \text{ needed to cover Ball rad } A\varepsilon \right)$

whats the dimension of a set of genomes for a given species?



$K \subset \bigcup U_\lambda$

Lebesgue covering lemma:

Suppose X is a metric space and $K \subset X$ is sequentially compact, then for every open covering of K $\{U_\lambda\}_{\lambda \in I}$ $\exists \delta > 0$ such that for every $k \in K$, $\exists \lambda \in I$ s.t.

$$B_\delta(k) \subset U_\lambda.$$

Main theorem for today: If X a metric space, then $K \subset X$ is compact if and only if it's sequentially compact.

Pf: Assume K compact, (x_n) a sequence in K
WTS $\exists$ convergent subsequence. i.e. $\exists x \in K$ s.t. sequence accumulates near x.

Assume (by contradiction) that $\forall x \in K$, $\exists \varepsilon_x > 0$ s.t. $x_n \notin B_{\varepsilon_x}(x)$ for n sufficiently large.

[i.e. suppose $\forall x \in K$, $\exists \varepsilon_x > 0$, N_x s.t. for $n \geq N_x$, $x_n \notin B_{\varepsilon_x}(x)$.]

Pf that is impossible:

Consider open sets $B_{\varepsilon_x}(x)$ for $x \in K$ these cover K.

Since K is compact, $\exists$ finite subcollection that covers i.e. pts $k_1, \ldots, k_r \in K$

s.t. $\bigcup_{i=1}^{r} B_{\varepsilon_{k_i}}(k_i)$ cover K.

$\Rightarrow (x_n) \subset K \subset \bigcup_{i=1}^{r} B_{\varepsilon_{k_i}}(k_i)$

$\Rightarrow \exists i$ s.t. $B_{\varepsilon_{k_i}}(k_i)$ contains infinitely many terms in my seq.

by hyp, $B_{\varepsilon_{k_i}}(k_i)$ only contains at most N_{k_i} terms

boom

$\Rightarrow \exists x \in K$ s.t. $\forall \varepsilon > 0$ $B_\varepsilon(x)$ contains ∞ly many terms in sequence (x_n).

$\Rightarrow \exists$ convergent subseq. (top pg 255 vol I)

Conversely, suppose K is sequentially compact

wts: K compact.

Consider some open cover $U_\lambda, \lambda \in I$ of K.

By Leb. CL cover has a lebesgue # $\delta > 0$.

suppose $\nexists$ finite subcovery.

construct a sequence as follows:

choose $x_1 \in K$ at random. then $B_\delta(x_1) \subset U_{\lambda_1}$

some $\lambda_1 \in I$.

But $U_{\lambda_1} \not\supset K$

so $\exists\, x_2 \in K \setminus U_{\lambda_1}$, choose $B_\delta(x_2) \subset U_{\lambda_2}$

keep going, at ith step

$$K \not\subset U_{\lambda_1} \cup \dots \cup U_{\lambda_i}$$

choose $x_{i+1} \in K \setminus (U_{\lambda_1} \cup \dots \cup U_{\lambda_i})$

$B_\delta(x_{i+1}) \subset U_{\lambda_{i+1}}$

notice $d(x_i, x_{i+j})$ $j > 0$

then by construction $x_{i+j} \notin U_{\lambda_1}, \dots, U_{\lambda_i}, \dots U_{\lambda_{i+j-1}}$

$x_{i+j} \notin U_{\lambda_i} \supset B_\delta(x_i)$

$\Rightarrow d(x_i, x_{i+j}) \geq \delta$

$\boxed{\subset = \subseteq}$

$\subsetneq \; \subseteq$

i.e. $\forall i \neq j \quad d(x_i, x_j) \geq \delta$.

$\Longrightarrow$ no convergent subsequence,

contradicting sequential compactness

! $\square$.