Recall: (last time)

If X is a metric space

~~potentially~~ ~~infinite~~ $S \subset X$ subset, ~~we~~ say S is compact

if whenever we have a collection of open sets $U_i \subset X$

$i \in I$, then $\exists\ J \subset I$ finite s.t. U_i, $i \in J$, also cover S.

which cover S,

Last time, we stated this theorem:

Thm: if X is a metric space, $S \subset X$ compact then S is closed and bounded.

and showed compact $\Longrightarrow$ bounded

Goal: if S is compact, then S is closed.

$C_r(a) = \{ b \in X \mid d(a,b) \le r \}$

$a \in X \quad r > 0$

Claim: $C_r(a)$ is closed in X

Pf: WTS, given $y \notin C_r(a)$ then $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(y) \subset X \setminus C_r(a)$

if $y \notin C_r(a) \Rightarrow d(a,y) > r$, let $\varepsilon < d(a,y) - r > 0$

if $d(y,b) < \varepsilon < d(a,y) - r$ then $d(a,y) \le d(a,b) + d(b,y) < d(a,b) + d(a,y) - r$

$\Rightarrow 0 < d(a,b) - r \Rightarrow d(a,b) > r$,

$\varepsilon < d-r$

$\Longrightarrow$ if $b \in B_\varepsilon(y)$ (i.e. $d(b,y) < \varepsilon$)

then $d(a,b) > r \Rightarrow b \notin C_r(a)$

i.e. $B_\varepsilon(y) \subset X \setminus C_r(a) \Leftrightarrow X \setminus C_r(a)$ is open

$\Longrightarrow C_r(a)$ closed $\square$

Goal: if S is compact, then S is closed.

$C_r(a) = \{ b \in X \mid d(a,b) \le r \}$

Pf of Goal: Suppose that S isn't closed

then $\exists\ x \in X \setminus S$ s.t. $x \notin (X \setminus S)^\circ$

i.e. $\forall r > 0$, $B_r(x) \not\subset (X \setminus S)$ i.e. $B_r(x) \cap S \neq \emptyset$

and therefore, $C_r(x) \cap S \supset B_r(x) \cap S \neq \emptyset$

Consider closed balls

$C_{1/n}(x)$ and the complements $U_n = X \setminus C_{1/n}(x)$ ← open

consider $\bigcup_n U_n = \bigcup_n (X \setminus C_{1/n}(x)) = X \setminus \left(\bigcap_n C_{1/n}(x)\right)$ de Morgan

but $\bigcap C_{1/n}(x) = \{x\}$

(if $y \in \bigcap C_{1/n}(x)$

$\Longrightarrow d(x,y) \le \frac{1}{n}$ all $n \Rightarrow d(x,y) = 0 \Rightarrow x = y$.)

Since $x \notin S \Rightarrow S \subset X \setminus \bigcap_n C_{1/n}(x) = X \setminus \{x\} = \bigcup U_n$

Goal: if S is compact, then S is closed.

(Assume S compact, and for sake of contradiction, not closed.) $x \in X \setminus S$, not in $(X \setminus S)^\circ$

$\Longrightarrow S \subset \bigcup U_n \Rightarrow S \subset$ finite union U_{n_i} $n_1, \dots, n_k$

$\Rightarrow S \subset U_N \quad N = \max\{n_1, \dots, n_k\}$

$S \subset \underbrace{(X \setminus C_{1/N}(x))}_{U_N} \Rightarrow C_{1/N}(x) \cap S = \emptyset$ contradiction

$B_{1/N}(x) \cap S \neq \emptyset$,

Lebesgue covering lemma

Suppose $K \subset X$ subset and U_λ $\lambda \in I$ open covering of K

we say that cover has Lebesgue number (mesh fineness)

of δ if $\forall k \in K\ \exists\ \lambda \in I$ s.t. $B_\delta(k) \subset U_\lambda$.

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

Definition $S \subset X$ is sequentially compact if any sequence (s_n) in S has a subsequence (s_{n_i}) which converges in S.

LC Lemma: If $K \subset X$ is sequentially compact then every covering of K has a Lebesgue number!