


Recall: (last time)  
 If  $X$  is a metric space  
 If  $S \subset X$  subset, ~~say~~  $S$  is compact  
 if whenever we have a collection of open sets  $U_i \subset X$   
 $i \in I$ , then  $\exists J \subset I$  finite s.t.  $U_i, i \in J$ , also  
 which cover  $S$ .

Last time, we stated this theorem:  
 If  $X$  is a metric space,  $S \subset X$  compact then  
 $S$  is closed and bounded  
 and showed compact  $\Rightarrow$  bounded



Goal: if  $S$  is compact, then  $S$  is closed.  
 $C_r(a) = \{b \in X \mid d(a,b) \leq r\}$   
 $a \in X, r > 0$   
 Claim:  $C_r(a)$  is closed in  $X$   
 PR: WTS given  $y \notin C_r(a)$  then  $\exists \epsilon > 0$  s.t.  $B_\epsilon(y) \cap C_r(a) = \emptyset$   
 if  $y \in C_r(a) \Rightarrow d(a,y) \leq r$ . Let  $\epsilon = d(a,y) - r > 0$   
 if  $d(y,b) \leq \epsilon < d(a,y) - r$  then  $d(a,y) \leq d(a,b) + d(b,y) < d(a,b) + d(a,y) - r$   
 $r < d(a,b) \Rightarrow b \notin C_r(a)$

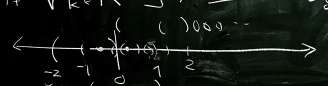
$\Rightarrow$  if  $b \in B_\epsilon(y)$  (i.e.  $d(b,y) \leq \epsilon$ )  
 then  $d(a,b) > r \Rightarrow b \notin C_r(a)$   
 i.e.  $B_\epsilon(y) \subset X \setminus C_r(a) \Rightarrow X \setminus C_r(a)$  is open  
 $\Rightarrow C_r(a)$  closed  $\square$

Goal: if  $S$  is compact, then  $S$  is closed.  
 $C_r(a) = \{b \in X \mid d(a,b) \leq r\}$   
 PR of Goal: Suppose that  $S$  is not closed.  
 then  $\exists x \in X \setminus S$  s.t.  $x \notin \overline{X \setminus S}$   
 i.e.  $\forall r > 0, B_r(x) \not\subset X \setminus S$  i.e.  $B_r(x) \cap S \neq \emptyset$   
 and therefore  $C_r(x) \cap S \neq \emptyset$

Consider closed balls  $C_{1/n}(x)$  and the complements  $U_n = X \setminus C_{1/n}(x)$   
 consider  $\bigcup_n U_n = \bigcup_n (X \setminus C_{1/n}(x)) = X \setminus \bigcap_n C_{1/n}(x)$   
 but  $\bigcap_n C_{1/n}(x) = \{x\}$   
 (if  $y \in C_{1/n}(x)$  then  $d(x,y) \leq 1/n$  for all  $n \Rightarrow d(x,y) = 0 \Rightarrow x=y$ )  
 $\Rightarrow \bigcup_n U_n = X \setminus \{x\}$  (de Morgan)

Goal: if  $S$  is compact, then  $S$  is closed.  
 (Assuming  $S$  compact, and for sake of contradiction, not closed)  
 $\Rightarrow S \subset \bigcup_n U_n \Rightarrow S \subset \text{finite union } U_{n_1}, \dots, U_{n_N}$   
 $\Rightarrow S \subset U_N, N = \max\{n_1, \dots, n_N\}$   
 $S \subset X \setminus C_{1/N}(x) \Rightarrow C_{1/N}(x) \cap S = \emptyset$  contradiction  
 $B_{1/N}(x) \cap S \neq \emptyset$

Lebesgue covering lemma  
 Suppose  $K \subset X$  subset and  $U_\lambda, \lambda \in I$  cover of  $K$   
 we say that cover has Lebesgue number (mesh fineness)  
 of  $S$  if  $\forall k \in K \exists \lambda \in I$  s.t.  $B_\delta(k) \subset U_\lambda$



Definition:  $S \subset X$  is sequentially compact if any sequence  $(s_n)$   
 in  $S$  has a subsequence  $(s_{n_k})$  which converges in  $S$ .  
 Lemma: If  $K \subset X$  is sequentially compact  
 then any cover of  $K$  has a Lebesgue number!