3) $f_n(x) = \sum_{i=0}^{n} x^i/i!$ on $[0,1]$

exam 2

show is Cauchy w/r/t $\|g\| = \int_0^1 |g(x)|dx$

WTS: given $\varepsilon > 0$ $\exists N > 0$ s.t. $n,m \geq N$ then $\|f_n(x) - f_m(x)\| \leq \varepsilon$.

examine this quantity:

given $N > 0$ $n,m \geq N$ $\|f_n(x) - f_m(x)\|$

$$= \left\| \sum_{i=0}^{n} x^i/i! - \sum_{i=0}^{m} x^i/i! \right\| < \varepsilon$$

WLOG assume $n > m$

$$= \left\| \sum_{i=m+1}^{n} x^i/i! \right\|$$

$$= \int_0^1 \left| \sum_{i=m+1}^{n} x^i/i! \right| dx$$

$$\leq \int_0^1 \sum |x^i/i!| \, dx$$

$$\leq \int_0^1 \sum_{i=m+1}^{n} |1/i!| \, dx$$

$$= \sum_{i=m+1}^{n} 1/i! \leq \sum_{i=N}^{n} 1/i! \leq \sum_{i=N}^{\infty} 1/i! < \varepsilon$$

various observations that $\sum_{i=0}^{\infty} 1/i!$ converges $= e$

$$\lim_{N\to\infty} \sum_{i=N}^{\infty} 1/i! = 0 \Rightarrow N >> 0$$

then $\sum 1/i! < \varepsilon$

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4) $\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{h} \mathbb{R}^3$

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "g"] \arrow[rr, bend right, "f"'] & \mathbb{R}^2 \arrow[r, "h"] & \mathbb{R}^3
\end{tikzcd}$$

if f^{-1} was cont diff.

$$(f^{-1})'(f(x)) = (f'(x))^{-1}$$

if inverse exists $\Longrightarrow$ $f'(x)$ is always invertible.

$f(x) = h(g(x))$ $\quad f'(x) = h'(g(x))\,g'(x)$

3x2 $\qquad$ 2x3

has a null space $\neq$ 0

so not invertible.

Advanced Calculus II, Fall 2022, Homework 9

Instructor: Danny Krashen

Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Suppose X and Y are metric spaces and (f_n) is a sequence of continuous functions. Show that if f_n converges uniformly to $f : X \to Y$, then f is also continuous.

2. Consider the sequence of functions described by $f_n(x) = e^x \sin(n)$ defined on the interval $[0, 1]$. Show that this sequence is uniformly equicontinuous: that is, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $n > 0$, whenever we have $|x_1 - x_2| < \delta$ with $x_1, x_2 \in [0, 1]$, then we have $|f_n(x_1) - f_n(x_2)| < \epsilon$.

Recall: f is uniformly cont if $\forall \varepsilon > 0\ \exists\ \delta > 0$ s.t.

$|x_0 - x_1| < \delta \Rightarrow |f(x_0) - f(x_1)| < \varepsilon \quad \forall x_0, x_1$

If (f_n) a sequence, it is uniformly equicontinuous if

$\forall \varepsilon > 0\ \exists\ \delta > 0$ s.t. $|x_0 - x_1| < \delta \Rightarrow |f_i(x_0) - f_i(x_1)| < \varepsilon$

all x_0, x_1, all i.

3. Recall that a sequence of real numbers (a_n) is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges. Let X be the set of convergent sequences.

   (a) (optional) Show that X is a normed vector space with respect to component-wise addition and scalar multiplication, and with the norm $||(a_n)|| = \sum_{n=1}^{\infty} |a_n|$.

   (b) Suppose we are given a sequence of elements in X. That is, we have a sequence (x_i) consisting of $x_1, x_2, \ldots$ with each x_i itself a sequence, say $x_i = (a_{i,n})$. Suppose that (x_i) is Cauchy with respect to the above norm.

      i. (required) Show that for each n, the sequence $(a_{i,n})$ is convergent (regarded as a sequence in the variable i).

      ii. (required) Write $a_n = \lim_{n\to\infty} a_{i,n}$. Show that (a_n) is absolutely convergent. Hint: use $|a_n| = |a_n - a_{i,n} + a_{i,n}| \leq |a_n - a_{i,n}| + |a_{i,n}|$ for every $i > 0$.

      iii. (required) Show that $\lim_{i\to\infty} x_i = (a_n)$. Hint: break up the sum $\sum\limits_{n=1}^{\infty} |a_{i,n} - a_n|$ into two sums $\sum\limits_{n=1}^{k} |a_{i,n} - a_n| + \sum\limits_{n=k+1}^{\infty} |a_{i,n} - a_n|$ and choose k to make the second summand small first.

4. Let X be the vector space of polynomial functions of the form $f(x) = ax^2 + bx + c$ from $[0,1]$ to $\mathbb{R}$. Consider the following norms on X:

   1. $||f||_s = \sup\{|f(x)| \mid x \in [0,1]\}$
   2. $||ax^2 + bx + c||_c = \max\{|a|, |b|, |c|\}$

   Show that for a sequence (f_n) in X, we have that $\lim_{n\to\infty} f_n = f$ with respect to the metric given by $||\ ||_s$ if and only if $\lim_{n\to\infty} f_n = f$ with respect to the metric given by $||\ ||_c$.

1 5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x\cos y, x\sin y)$. Does f have a differentiable inverse near $(0,1)$? That is, is possible to find open sets $U, V \subset \mathbb{R}^2$ with $(0,1) \in U$ with $f(U) = V$ and a differentiable function $g : V \to U$ such that $fg = id = gf$?

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (e^x\cos y, e^x\sin y)$. Does f have a differentiable inverse near $(0,1)$? That is, is possible to find open sets $U, V \subset \mathbb{R}^2$ with $(0,1) \in U$ with $f(U) = V$ and a differentiable function $g : V \to U$ such that $fg = id = gf$?

7. As in problem 3 let X be the set of absolutely convergent sequences with the norm given by $||(a_n)|| = \sum_{n=1}^{\infty} |a_n|$ Consider the linear transformation $T : X \to X$ given by $T(a_n)$ is the sequence (b_n) where $b_n = a_{n+1}$. Show that $||T|| = 1$ (the operator norm).

2 8. As in problem 3 let X be the set of absolutely convergent sequences with the norm given by $||(a_n)|| = \sum_{n=1}^{\infty} |a_n|$ Consider the function $f : X \to X$ given by $f(a_n)$ is the sequence $(1/n\ a_n)$. Is f continuous? Is it differentiable? (this is a bit of a trick question).

2 9. As in problem 3 let X be the set of absolutely convergent sequences with the norm given by $||(a_n)|| = \sum_{n=1}^{\infty} |a_n|$ Consider the function $f : X \to X$ given by $f(a_n)$ is the sequence (b_n) where $b_1 = a_1a_2$ and $b_n = a_n$ for $n > 1$.

   Fix an absolutely convergent sequence (c_n).

   Let $T : X \to X$ be given by $T(a_n) = (c_2a_1, c_1a_2 + a_2, a_3, a_4, a_5, \ldots)$.

   (a) Show that T is a linear transformation.

   (b) Show that $T = f'(a_n)$.

10. Suppose $f : X \to Y$ is a map between metric spaces such that there exists some real number $L > 0$ such that $d(f(x), f(y)) \le Ld(x,y)$. Show that f must be continuous.

11. Suppose $T : X \to Y$ is linear transformation between normed linear spaces which is bounded. Show that T must be continuous.

12. Suppose (T_n) is a sequence of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^5$. Suppose that $\lim_{n\to\infty} T_n = T$ with respect to the operator norm. Show that for every $v \in \mathbb{R}^3$, we have $\lim_{n\to\infty} T_n(v) = T(v)$ in $\mathbb{R}^5$.

13. Consider the set $X = \{1/n | n = 1, 2, \ldots\}$. Define $d(x,y) = |x-y|$.

    (a) Is X a metric space with respect to this metric?

    (b) Is X complete?

$n\to\infty$

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x \cos y, x \sin y)$. Does f have a differentiable inverse near $(0, 1)$? That is, is possible to find open sets $U, V \subset \mathbb{R}^2$ with $(0, 1) \in U$ with $f(U) = V$ and a differentiable function $g : V \to U$ such that $fg = id = gf$?

$$f(f^{-1}(x)) = x \longrightarrow (f \circ f^{-1})'(x) = (Id)'(x)$$
$$= Id$$

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = Id$$

$$(f^{-1}(f(x)))' = (f^{-1})'(f(x))\, f'(x) = Id$$

if f^{-1} exists and is differentiable at $f(0,1)$

then $f'(0,1)$ is invertible

$$\underbrace{(f^{-1})'(f(0,1))}_{\text{inv.}} \cdot \underbrace{f'(0,1)}_{\text{invertible}} = id$$

$$\begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{bmatrix}$$

$$\det = x\cos^2 y + x\sin^2 y$$
$$= x(\cos^2 y + \sin^2 y) = x$$

$\det(f'(0,1)) = 0$ not invertible.

9. As in problem 3 let X be the set of absolutely convergent sequences with the norm given by $\|(a_n)\| = \sum_{n=1}^{\infty} a_n$. Consider the function $f : X \to X$ given by $f(a_n)$ is the sequence (b_n) where $b_1 = a_1 a_2$ and $b_n = a_n$ for $n > 1$.

Fix an absolutely convergent sequence (c_n).

Let $T : X \to X$ be given by $T(a_n) = (c_2 a_1, c_1 a_2 + a_2, a_3, a_4, a_5, \ldots)$.

(a) Show that T is a linear transformation.

(b) Show that $T = f'(a_n)$.

Squint: just look at $f(a)$ 2 variables $\mathbb{R}^2 - \{(a_1, a_2)\}$

$$f(a_1, a_2) = (a_1 a_2, a_2)$$

$$f(x,y) = (xy, y)$$

$$f'(x,y) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial \ldots & \end{bmatrix} = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$$

if $T : X \to Y$ is a linear transformation which is bounded.

then $T'(x) = T$ $S = T'(x)$ means

$$\lim_{h \to 0} \frac{\|T(x+h) - T(x) - S(h)\|}{\|h\|} = 0$$

$T(x) + T(h)$

$S = T$?

$$\frac{\|T(x+h) - T(x) - T(h)\|}{\|h\|}$$