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Solve some PDE $\rightsquigarrow \sum_n a_n e^{nt} = f(t)$

$$\sum a_n n e^{nt} = f'(t)$$

Q: When do these sums converge?

Q: when they converge, when are they continuous / differentiable?

Q: when can one take derivatives / integrals termwise?

Basic thing that makes this work: uniform convergence.

Reminder from last time:

Let (f_n) sequence of functions in $\mathrm{Fun}(X, Y)$ where Y a metric space, we say f_n converge uniformly to f if $\forall \varepsilon > 0 \; \exists N > 0$ s.t. if $n \geq N$ $d(f(x), f_n(x)) \leq \varepsilon$ for all $x \in X$.

[similar def: if $\forall x \in X, \forall \varepsilon > 0 \; \exists N$ s.t. $n \geq N$, $d(f(x), f_n(x)) \leq \varepsilon$] pointwise convergence.

"uniform metric" on $\mathrm{Fun}(X, Y)$: $d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$

Prop 11.2.7 Let X, Y metric spaces, Y complete. (f_n) sequence in $\mathrm{Fun}(X, Y)$ s.t. f_n conv. unif. to f

Let (x_k) be a sequence in X and $\lim_{k \to \infty} x_k = x$

and suppose $\lim_{k\to\infty} f_n(x_k)$ exist for all k

then $$\lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k) = \lim_{n\to\infty} \lim_{k\to\infty} f_n(x_k)$$

(all limits exist)

<u>Cor:</u> If f_n's are continuous and $f_n \to f$ uniformly then f continuous.

Given a series $$\sum_{k=0}^{\infty} f_k(x) = \lim_{n\to\infty} \sum_{k=0}^{n} f_k(x) = \lim_{n\to\infty} s_n(x)$$

define $$s_n(x) = \sum_{k=0}^{n} f_k(x)$$

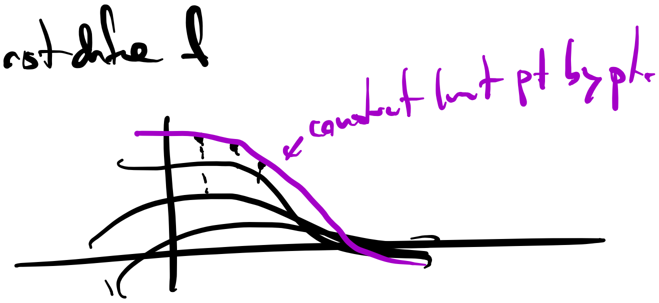
<u>Back up for a bit:</u>

we defined uniform convergence need <u>Cauchy</u>

<u>Def</u> X set Y metric space (f_n) sequence in $\mathrm{Fun}(X,Y)$ we say (f_n) is <u>uniformly Cauchy</u> if $\forall \varepsilon>0\ \exists N>0$ s.t. $n,m \geq N \Rightarrow d(f_n(x), f_m(x)) < \varepsilon$ all $x \in X$.

<u>Proposition:</u> If X set Y <u>complete</u> metric space, (f_n) uniformly Cauchy sequence in $\mathrm{Fun}(X,Y) \Rightarrow f_n \to f$ uniformly for some $f \in \mathrm{Fun}(X,Y)$

Pf: First define f



Define f by $f(x) = \lim_{n\to\infty} f_n(x)$ which exists

Why? because $(f_n(x))$ sequence in Y is Cauchy!

Why? $\forall \varepsilon > 0$ WTS $\exists N$ s.t. $n,m \geq N \Rightarrow d(f_n(x), f_m(x)) < \varepsilon$

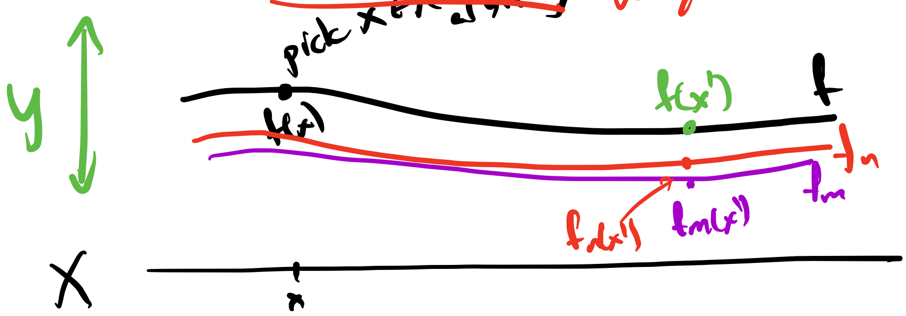
but know since (f_n) unif. Cauchy

$\exists N$ s.t. $n,m \geq N$, $d(f_n(x'), f_m(x')) < \varepsilon$ all $x' \in X$.

So $f_n(x)$'s are Cauchy, so they converge, can define $f(x) = \lim f_n(x)$

Claim $(f_n) \to f$ uniformly.

~~pick $x \in X$ arbitrary~~ why? didn't need to



choose $N \gg$ s.t. $f_n(x)$ close to $f(x)$ for $n > N$.

Given $\varepsilon > 0$
want $N > 0$ s.t.
$n \geq N \Rightarrow d(f_n(x'), f(x')) < \varepsilon$ all $x' \in X$.

and $N \gg$ so that f_n close to f_m unif. for $n,m \geq N$

x arbitrary.

Pick x'.

choose $N' > 0$ s.t. $n \geq N' \Rightarrow d(f(x), f_n(x)) \leq \frac{\varepsilon}{3}$

choose $N'' > 0$ s.t. $n, m \geq N'' \Rightarrow d(f_n(x'), f_m(x')) \leq \frac{\varepsilon}{3}$

all $x' \in X$

(uniform cauchy) let $N \geq N', N''$

if $n \geq N$

$d(f_n(x'), f(x'))$

$d(f_n(x'), f_m(x')) \leq \frac{\varepsilon}{3}$ all m also $\geq N$

since $\lim_{m \to \infty} f_m(x') = f(x')$ $m \gg M_{x'}$

then $d(f_m(x'), f(x')) < \frac{\varepsilon}{3}$

$$d(f_n(x'), f(x')) \leq d(f_n(x'), f_m(x')) + d(f_m(x'), f(x'))$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \quad \square$$

Given $\sum_{k=0}^{\infty} f_k(x) = \lim_{n \to \infty} \sum_{k=0}^{n} f_k(x) = \lim_{n \to \infty} s_n(x)$

$$\text{"} d(s_n, s_m) \text{"} = \sup \{ d(s_n(x), s_m(x)) \mid x \in X \}$$
$$= \sup \{ |s_n(x) - s_m(x)| \mid x \in X \}$$

$f: X \to Y$

$\mathbb{R}$

$$(\text{if } n \geq m) = \sup \{ |\sum_{k=m+1}^{n} f_k(x)| \mid x \in X \}$$