Almost done w/ Chap 8
- couple words today

Chapter 11 Spaces of functions

Up till now

Metric spaces
(limits, continuity
compactness / completeness) ⟶ Normed vector spaces
specific example
of metric spaces
(differentiation)

Function spaces

functions $X \to Y$ — X, Y are metric spaces
Y normed vector space...

$\mathrm{Fun}(X, Y)$
"vector space of functions"

Moral Justification

Fourier Series
$f(x) = \sum_{i=0}^{\infty} a_i f_i(x)$

Problem (λ)
$f^\lambda(x) = \sum_{n=0}^{\infty} g_n^\lambda(x)$

main tool: uniform convergence

Prop 8.46.

A function $f: U \to \mathbb{R}^n$ is continuously differentiable
$U \subset \mathbb{R}^n$

if and only if all its partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous.

Our starting point: only need Y to a metric space, not X.

Let X, Y metric spaces

Consider $\mathrm{Fun}(X,Y) = \{f : X \to Y\}$ all functions.

Define $d(f,g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}$

d_u uniform

d_∞ ∞

ex: $X = Y = \mathbb{R}$ $f = 0$ $g(x) = x$

$d(f,g) = \sup\{|x| \mid x \in \mathbb{R}\} = \infty$

on the other hand

- $d(f,g) = 0 \iff f = g$
- $d(f,g) = d(g,f)$

$$\overset{=}{\sup}\{d(f(x),g(x)) \mid x\in X\}$$

$$d(g(x),f(x))$$

$? \le \infty$ ✓

$\infty + \infty = \infty$

- $d(f,h) \le d(f,g) + d(g,h)$ ✓

$$\overset{=}{\sup}\{d(f(x),h(x)) \mid x\in X\}$$

for all x, $d(f(x),h(x)) \le d(f(x),g(x)) + d(g(x),h(x))$

Δ ineq

$$\le \sup\{d(f(x),g(x)) + d(g(x),h(x)) \mid x\in X\}$$

sup thy

$$\le \sup\{d(f(x),g(x)) \mid x\in X\} + \sup\{d(g(x),h(x)) \mid x\in X\}$$

$$= d(f,g) + d(g,h)$$

$$\sup\{d(f(x),g(x)) + d(g(x),h(x)) \mid x\in X\}$$

$$= \sup\{d(f(x),g(x)) + d(g(y),h(y)) \mid x,y\in X \text{ and } x=y\}$$

$$\le \sup\{d(f(x),g(x)) + d(g(y),h(y)) \mid x,y\in X\}$$

$$= \sup\{d(f(x),g(x)) \mid x\in X\} + \sup\{d(g(y),h(y)) \mid y\in X\}$$

<u>Def</u> We say that a sequence of functions $f_n \in \mathrm{Fun}(X,Y)$ <u>uniformly converges</u> to $f \in \mathrm{Fun}(X,Y)$ if $\forall \varepsilon > 0$ $\exists N$ s.t. $n \ge N$ then $d(f_n,f) < \varepsilon$.

Def (f_n) in $Fun(X,Y)$ is uniformly Cauchy
if $\forall \varepsilon > 0\ \exists N$ s.t. if $n,m \geq N$ then
$d(f_n, f_m) < \varepsilon$.

Remark if we define $BFun(X,Y) = \{ f : X \to Y \mid f$ is bounded $\}$
(we say f is bounded if $f(X) = \{f(x) \mid x \in X\}$ is bounded in Y)

d is a metric on $BFun(X,Y)$

because if $f, g \in BFun(X,Y)$ then $d(f,g) < \infty$

recall: $f(X)$ bounded $\Rightarrow \forall y \in Y\ \exists R > 0$ s.t.
$f(X) \subset B_{R''}(y)$

$g(X) \subset B_{R'}(y)$ choose $R = \max\{R'', R'\}$

let $h(x) = y$ all x
const. function

$d(f,g) \leq d(f,h) + d(h,g) \leq 2R$

Limits of sequences of functions

Prop If X is a set, Y metric space, for a seq in $\mathrm{Fun}(X, Y)$

- (f_n) converges uniformly $\Rightarrow$ it is uniformly Cauchy

Conversely, if Y is complete

then (f_n) unif. cauchy $\Rightarrow$ converges uniformly.

Prop Let X, Y be metric spaces w/ Y complete

(f_n) sequence in $\mathrm{Fun}(X, Y)$ which conv. uniformly to f

and (x_k) seq in X which converges and such that

$\lim_{k\to\infty} f_n(x_k)$ converges also. then all n

$$\lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k) = \underbrace{\lim_{n\to\infty} \lim_{k\to\infty} f_n(x_k)}_{\text{and this converges.}}$$

Cor if (f_n) are a seq of continuous functions s.t.

f_n converges to f uniformly then f is also continuous.

Proof ~~using prop~~:

f continuous $\Leftrightarrow$ whenever $\lim_{k\to\infty} x_k = x$, $\lim_{k\to\infty} f(x_k) = f(x)$

Suppose $\lim_{k\to\infty} x_k = x$

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k)$$

$$\text{prop} = \lim_{n\to\infty} \underbrace{\lim_{k\to\infty} f_n(x_k)}_{f_n \text{ cont}} = \lim_{n\to\infty} f_n(x) = f(x)$$

usg $d(f_n(x_k), f(x_k)) \leq \partial(f_n, f)$

$$\Rightarrow \left(\lim_{\substack{n\to\infty \\ \text{unif.}}} f_n = f \Rightarrow \lim_{n\to\infty} f_n(x_k) = f(x_k) \right)$$