Last time we almost got to the inverse function thm.

Recall:

<u>Thm:</u> Let $f: \underset{\mathbb{R}^n}{U} \longrightarrow \mathbb{R}^n$ be continuously differentiable

and suppose $p \in U$ s.t. $f'(p)$ is an invertible lin. trans.

then $\exists B = B_\varepsilon(p) \subset U$ such that if we set $W = f(B)$

we have $f: B \to W$ is bijective & the inverse function

is also cont. diff. w/ $(f^{-1})'(f(x)) = (f'(x))^{-1}$

for $x \in B$.

(from chain rule: $f^{-1}(f(x)) = id$

$$(f^{-1})'(f(x)) \cdot f'(x) = (id)' = I_n$$

$id: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$(x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_n)$

$$(\partial x_i / \partial x_j) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$\cdot (f'(x))^{-1}$ $\quad (f'(x))^{-1}$

$$(f^{-1})' f(x) \cancel{f'(x) f'(x)^{-1}} = f'(x)^{-1}$$

<u>Today:</u> we'll show f is injective for sufficiently small ε.

i.e. want to show

$$B = B_\varepsilon(p) \xrightarrow{f} \mathbb{R}^n$$

Crazy strategy: $f'(p)$ invertible

Chose $x_0 \in U$ (close to p)

Consider $\varphi_{x_0}(x) = x + f'(p)^{-1}(f(x_0) - f(x))$

Notice that $\varphi(x) = x$ only when

$$f'(p)^{-1}(f(x_0) - f(x)) = 0$$

only when $f(x_0) - f(x) = 0$

$$f(x_0) = f(x)$$

Goal: (to show 1-1) want ε s.t. if $x, x_0 \in B_\varepsilon(p)$

then $f(x_0) = f(x)$ only when $x_0 = x$

$\Updownarrow$

$\varphi(x) = x$

way we'll find ε: we'll show if ε small

then $\|\varphi_{x_0}'(x)\| < \frac{1}{2}$ in this case for all x

MVT: for $x \in B_\varepsilon(p)$

$$\|\varphi(x) - \varphi(x_0)\| \le \frac{1}{2}\|x - x_0\|$$

but notice $\varphi(x) = x \iff f(x) = f(x_0)$

so if $f(x) = f(x_0)$ then $\varphi(x) = x$

also $f(x_0) = f(x_0) \implies \varphi(x_0) = x_0$

$$\|x - x_0\| \le \frac{1}{2}\|x - x_0\|$$

$$\implies \|x - x_0\| = 0 \implies x = x_0$$

we showed: if we knew that $\forall x \in B_\varepsilon(p)$

that $\|\varphi'_{x_0}(x)\| \le \frac{1}{2}$ $\quad \varphi_{x_0}(x) = x + f'(p)^{-1}(f(x_0) - f(x))$

then $f(x) = f(x_0) \iff x = x_0$. for $x, x_0 \in B_\varepsilon(p)$

Notation: $A = f'(p)$ lin trans (const)

$y = f(x_0)$ (const vector)

$$\varphi(x) = x + A^{-1}(y - f(x)) = x + A^{-1}y - A^{-1}f(x)$$

$$\varphi'(x) = I_n - A^{-1}f'(x) = A^{-1}(A - f'(x))$$

(SKIP AHEAD)

<u>Sidebar:</u>

if $g : \mathbb{R}^n \to \mathbb{R}^m$ $\quad$ $T : \mathbb{R}^m \to \mathbb{R}^\ell$

some fcn $\quad$ lin trans

$(Tg)'$ $\quad$ assume g is diff

T as a function is differentiable $\mathbb{R}^m \to \mathbb{R}^\ell$

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} & \cdots & T_{1m} \\ \vdots & & & & \vdots \\ T_{\ell 1} & - & - & - & T_{\ell m} \end{bmatrix}$$

$$T(x_1, \ldots, x_m) = (T_1(\vec{x}), T_2(\vec{x}), \ldots, T_\ell(\vec{x}))$$

$$T_i(x_1, \ldots, x_m) = T_{i1}x_1 + T_{i2}x_2 + \cdots + T_{im}x_m$$

$$T' = \begin{bmatrix} \partial T_1/\partial x_1 & \partial T_1/\partial x_2 & \cdots & \partial T_1/\partial x_m \\ \vdots & & & \\ \partial T_\ell/\partial x_1 & - & - & \end{bmatrix}$$

$$\partial T_i/\partial x_j = T_{ij}$$

$$T' = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1m} \\ & & & \end{bmatrix}$$

$$T : \mathbb{R}^m \longrightarrow \mathbb{R}^\ell$$

$$T' : \mathbb{R}^m \longrightarrow L(\mathbb{R}^m, \mathbb{R}^\ell) \quad \text{"slopes"}$$

$$x \longmapsto T$$

if $g : \mathbb{R}^n \to \mathbb{R}^m$ $\qquad$ $T : \mathbb{R}^m \to \mathbb{R}^\ell$

same fcn $\qquad$ lnr trans

assume g is diff

$$(Tg)'(x)$$
$$\shortparallel$$
$$T'(g(x)) \cdot g'(x)$$
$$T(g'(x))$$

$(\lambda f)' = \lambda f'$

$$\varphi'(x) = I_n - A^{-1} f'(x) = A^{-1}(A - f'(x))$$

$$\begin{aligned}
\|\varphi'(x)\| &= \| A^{-1} (A - f'(x)) \| \\
&\leq \|A^{-1}\| \ \|A - f'(x)\| \\
&= \|A^{-1}\| \ \|f'(p) - f'(x)\|
\end{aligned}$$

$$\left(\begin{array}{l}
\mathbb{R}^n \longrightarrow L(\mathbb{R}^n,\mathbb{R}^m) \longrightarrow \mathbb{R} \\
x \xmapsto{\text{cont}} f'(x) \\
\qquad\qquad\qquad T \longmapsto \|T - f'(p)\| \\
x \longmapsto \|f'(x) - f'(p)\| \\
\quad \underline{\text{continuous}} \qquad = \|f'(p) - f'(x)\| \\
\text{and sends } p \longmapsto 0
\end{array}\right)$$

$\exists\, \varepsilon > 0$ s.t. $\|x-p\| < \varepsilon$ $\quad \|f'(p) - f'(x)\| < \frac{1}{2}$

Since f is cont. diff at $p \Rightarrow$

$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ s.t. $\|x-p\| < \delta \Rightarrow \|f'(p) - f'(x)\| < \varepsilon$

$\exists\, \delta > 0$ s.t. $\forall\, x \in B_\delta(p) \quad \|f'(p) - f'(x)\| < \varepsilon$

want $\|\varphi'(x)\| < \frac{1}{2}$

$$\leq \|A^{-1}\|\, \|f'(p) - f'(x)\| \overset{\text{want}}{\leq} \frac{1}{2}$$

$$\|f'(p) - f'(x)\| \leq \frac{1}{2\|A^{-1}\|}$$

choose $\varepsilon = \dfrac{1}{2\|A^{-1}\|}$ above

$$\exists\, \delta \text{ s.t. } x \in B_\delta(p) \Rightarrow \|f'(p) - f'(x)\| \leq \frac{1}{2\|A^{-1}\|}$$

$$\Rightarrow \|\varphi'(x)\| \leq \frac{1}{2} \quad \square$$

Implicit function theorem

Idea: consider a lin trans.

$$\mathbb{R}^{n+m} \xrightarrow{T} \mathbb{R}^m$$

m conditions on vectors,

$$T(\vec{v}) = 0$$

$$\| $$

$$(T_1(\vec{v}), T_2(\vec{v}), \ldots, T_m(\vec{v}))$$

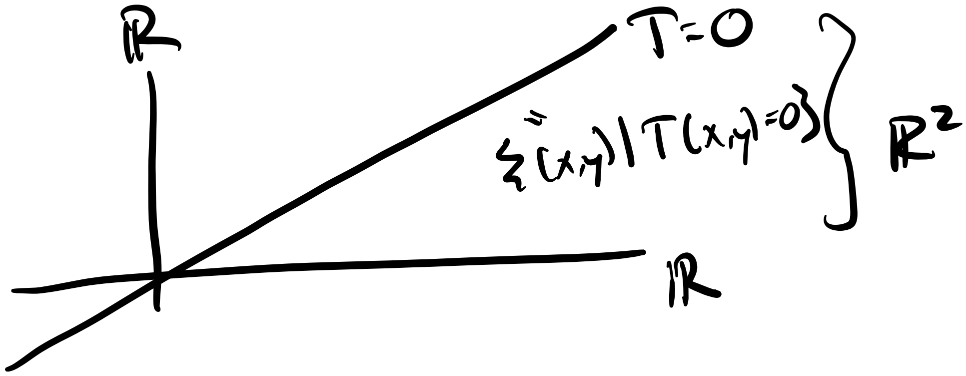
if you're lucky, have m indep. conditions
sol'n's are $n+m-m = n$ dim'l.

if luckier, solns determined by first n coordinates.

"Thm": if $T : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$

$$\overbrace{\Big[\; \underbrace{\begin{matrix} T_1 \\ m\times n \end{matrix}}_{n} \;\Big|\; \underbrace{\begin{matrix} T_2 \\ m\times m \end{matrix}}_{m} \;\Big]}^{n+m} \Big\}\, m \qquad \text{if } T_2 \text{ is invertible}$$

then for any $x \in \mathbb{R}^n$ $\exists$ unique $y \in \mathbb{R}^m$ such that $T(x_1, \dots, x_n, y_1, \dots, y_m) = 0$



$$\mathbb{R}^2 \xrightarrow{T} \mathbb{R}$$

Thm (Implicit function thm)

if $f: \underset{\mathbb{R}^{n+m}}{\underset{\cap}{U}} \longrightarrow \mathbb{R}^m$ cont. diff.

and $p \in U$ w/ $f'(p): \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$

having the form $\left[\underset{m \times n}{A} \mid \underset{m \times m}{B}\right]$

s.t. B is invertible then $\exists\, \varepsilon > 0$ s.t. for

$(x,y) \in B_\varepsilon(p)$ w/ $f(x,y) = 0$

y is uniquely determined by x

i.e. $y = g(x)$ g cont. diff.