"Last time"

Def of the Fréchet derivative of a function $f: U \to Y$, $U \subset X$

$f'(x)$ X, Y metric spaces

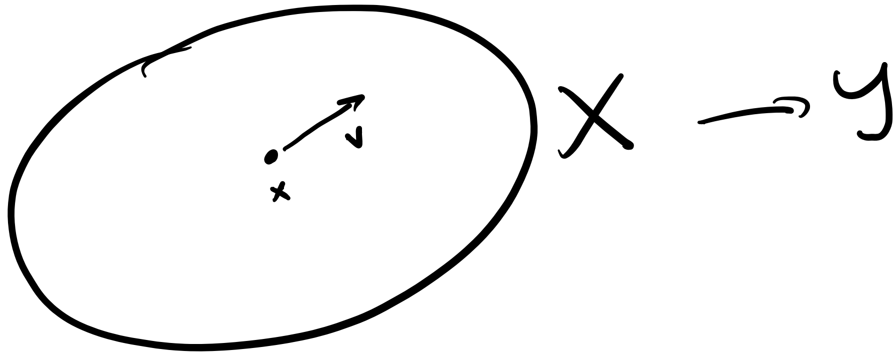
Def of the Gateaux differential $f: U \to Y$, $U \subset X$

$df(x, v)$ ← at x in v direction

(partial derivative) $X = \mathbb{R}^n$

how f changes when we move from x to $x + \varepsilon v$

$\frac{df}{dx}$ $\frac{d}{dx}$ dx df



$$df(x, v) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon v) - f(x)}{\varepsilon} \in Y$$

$X = \mathbb{R}^2$ $Y = \mathbb{R}^3$

$$f(\vec{x}) = (e^{x_1+x_2},\ x_1 - x_2,\ x_1 x_2^2 + x_2)$$

$$\phantom{f(\vec{x})} \shortparallel$$

$$f(x_1, x_2)$$

$x_1 = 0 \qquad x_2 = 1 \qquad v = (1,0)$

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon v) - f(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{f((0,1) + \varepsilon(1,0)) - f(0,1)}{\varepsilon}$$

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon, 1) - f(0,1)}{\varepsilon}$$

$$\lim_{\varepsilon \to 0} \frac{(e^{\varepsilon+1}, \varepsilon - 1, \varepsilon + 1) - (e^{1}, -1, 1)}{\varepsilon}$$

$$\lim_{\varepsilon \to 0} \frac{(e^{\varepsilon+1} - e, \varepsilon, \varepsilon)}{\varepsilon}$$

$$\lim_{\varepsilon \to 0} \frac{e^{\varepsilon+1} - e}{\varepsilon} \qquad \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} = 1 \qquad \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} = 1$$

} L'hopital's rule

$$\lim_{\varepsilon \to 0} \frac{e^{\varepsilon+1} \cdot 1 - 0}{1} = e \qquad df(x,v) = (e, 1, 1)$$

at direction

$$df(x,v) = df((0,1),(1,0)) = (e,1,1)$$

definition of the partial derivative

$$= \frac{\partial f}{\partial x_1}\bigg|_{(0,1)}$$

$(0,1) + \varepsilon(1,0)$

$(\varepsilon, 1)$

$$\boxed{x_1 \longleftrightarrow (1,0)}$$

Interpretation:

$$f(x+\varepsilon v) \approx f(x) + \frac{df(x,v)}{\|v\|}\varepsilon \qquad \varepsilon \text{ small}$$

$$f((0,1) + (1,0)\varepsilon) \approx f(0,1) + (e,1,1)\varepsilon$$

Contrast: Fréchet derivative $f'(x)$

$$f(x+\varepsilon v) \approx f(x) + f'(x)v\varepsilon$$

$$f'(x)v = \frac{df(x,v)}{\|v\|}$$

$$df(x,\lambda v) = \lambda\, df(x,v)$$

$$df(x,v) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon v) - f(x)}{\varepsilon}$$

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon v) - \left(f(x) + df(x,v)\varepsilon\right)}{\varepsilon} = 0$$

$$\lim_{\|h\| \to 0} \frac{\| f(x+h) - (f(x) + f'(x)h) \|}{\|h\|} = 0$$

$h = \varepsilon v$ $\quad f'(x)h$

$f'(x) v \varepsilon$