Today: Normed vector spaces
(combination of metric spaces & vector spaces)

all vector spaces will be vector spaces over $\mathbb{R}$.

Given a vector space $V$, want a distance function

$$d : V \times V \longrightarrow \mathbb{R}$$

nice properties we might want:

- $d(\lambda v, \lambda w) = |\lambda|\, d(v,w)$
- $d(v+u, w+u) = d(v,w)$

Notice in this case, $d(v,w) = d(v-v, w-v)$
$= d(0, w-v)$
"length of $w-v$"

so distance function is determined by distances from $0$ ("lengths")

notation $d(0,v) = \|v\|$ (norm)

<u>Def</u> A normed vector space

Is a vector space $V$ together w/ a function

$$\|\cdot\| : V \longrightarrow \mathbb{R}$$
$$v \longmapsto \|v\| \text{ such that}$$

1) $\|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = 0$

2) $\|cv\| = |c|\,\|v\|$ $\quad c \in \mathbb{R}$

3) $\|v+w\| \leq \|v\| + \|w\|$ triangle.

Observe: If $V$ is a normed vector space then it is also a metric space by defining

$$d(v,w) = \|v-w\|$$

Check: • $d(v,w) \geq 0$ since $d(v,w) = \|v-w\| \underset{(1)}{\geq} 0$

• $d(v,w) = 0 \iff \|v-w\| = 0 \underset{(1)}{\iff} v-w=0$
$v=w$

• $d(v,w) = \|v-w\| = |-1|\,\|v-w\| \overset{(2)}{=} \|(-1)(v-w)\| = \|w-v\| = d(w,v)$

• $d(v,u) \leq d(v,w) + d(w,u)$

since

$$\|v-u\| = \|(v-w)+(w-u)\| \underset{(3)}{\leq} \|v-w\| + \|w-u\| = d(v,w) + d(w,u)$$

Conversely, given distance function

$d: V\times V \longrightarrow \mathbb{R}$ such that

- $d(v+u, w+u) = d(v,w)$ &
- $d(\lambda v, \lambda w) = |\lambda| d(v,w)$

then $\|v\| = d(0,v)$ is a norm on $V$,

<u>example:</u> $V = \mathbb{R}^n$ $\quad d(v,w) = \sqrt{\sum_{i=1}^{n} (v_i - w_i)^2}$

has properties

$\Rightarrow \|v\| = d(0,v) = \sqrt{\sum v_i^2}$ is a norm on $\mathbb{R}^n$

standard Euclidean norm.

<u>ex:</u> $d(v,w) = \max\{|v_i - w_i| \mid i = 1, \dots, n\}$

is also a distance, gives a norm

$\|v\| = \max\{|v_i|\}$

<u>Notation:</u>

$X, Y$ vector spaces, $L(X,Y) = \{$ linear trans. from $X$ to $Y \}$

$$L(X) = L(X, X)$$

$$L(\mathbb{R}^n) = M_n(\mathbb{R})$$

$$L(\mathbb{R}^n, \mathbb{R}^m) = M_{m,n}(\mathbb{R})$$

$(a_{ij}) \rightsquigarrow$ lin. trans.

$(a_{ij}) \rightsquigarrow$ trans

Given $(a_{ij}) \longmapsto \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$= \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ \vdots \\ a_{m1}v_1 + \cdots + a_{mn}v_n \end{bmatrix}$$

$(a_{ij})$ is the lin tran taking

$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ to $\begin{bmatrix} a_{11}v_1 + \cdots \\ \\ \end{bmatrix}$

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th place}$$

$$\text{Secret } T \longleftrightarrow (a_{ij})$$

$$Tv \longleftrightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v \end{bmatrix}$$

$$Te_j \longleftrightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

$a_{ij} = i$th entry of $Te_j$

$$v \cdot w = \underbrace{v^t w}_{\text{matrix mult.}}$$

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \qquad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$T \longmapsto \begin{bmatrix} \text{---}\ r_1\ \text{---} \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} r_1 \cdot v \\ r_2 \cdot v \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix} \begin{array}{l} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow j^{th} \\ \\ \\ \end{array}$$

$$= c_j$$

# Derivatives

Given $f : U \to \mathbb{R}^m$

$U$ open $\subset \mathbb{R}^n$

what's the derivative $f'$ ?

$m = n = 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$x \in U$

alternately want to say $f(x+h) \approx f(x) + x f'(x)$

i.e. $f'(x)$ is the number $T$ s.t.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = T$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - T = 0$$

$$\lim_{h \to 0} \frac{f(x+h) - (f(x) + Th)}{h} = 0$$

in general, the derivative of $f: U \to \mathbb{R}^m$

is a linear approx of $f$ at $x$.

i.e. some $T \in L(\mathbb{R}^n, \mathbb{R}^m)$

s.t. $f(x+h) \approx f(x) + Th$

i.e. $f'(x) = T$ s.t.

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - (f(x) + Th)\|}{\|h\|} = 0$$