Advanced Calculus II, Fall 2022, Homework 1

Instructor: Danny Krashen

Discussing the problems with other people is encouraged, but you must write up your own work independently!

1. (required) Show that if X is the set of continuous functions from the interval $[0, 1]$ to the real line $\mathbb{R}$, then setting $d(f, g) = \int_0^1 |f(x) - g(x)| dx$ defines a metric on X.

2. (required) Show that for $X = \mathbb{R}^2$, we can define a metric by $d(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

3. (required) (Libl, Exercise 7.3.2)

(a) Show that $d(x, y) = \min\{1, |x - y|\}$ defines a metric on $\mathbb{R}$.

(b) Show that a sequence converges in $(\mathbb{R}, d)$ if and only if it converges in the standard metric.

(c) Find a bounded sequence in $(\mathbb{R}, d)$ that contains no convergent subsequence.

4. (required) Consider the function $\phi(t) = \frac{t}{t+1}$.
   (a) Show (without using calculus!) that $\phi(t)$ is monotonically increasing – that is, $t_1 > t_2$ implies $\phi(t_1) \geq \phi(t_2)$.

   (b) Show that $\phi(t_1 + t_2) \leq \phi(t_1) + \phi(t_2)$ for $t_1, t_2 \geq 0$.

   (c) Show that if d is a metric on a set X, then we get a new metric $\overline{d}$ by defining $\overline{d}(x, y) = \phi(d(x, y))$.

5. (optional) Show that for $X = \mathbb{R}^2$, we can define a metric by $d(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2|$.

6. (optional) Let X be the set of lines in the plane through the origin. For lines L_1, L_2, let $\angle(L_1, L_2)$ denote the angle from L_1 to L_2.

   (a) If we set $d(L_1, L_2) = |\angle(L_1, L_2)|$, does this define a metric? Why or why not?

   (b) If we set

$$d(L_1, L_2) = \begin{cases} |\angle(L_1, L_2)| & \text{if } |\angle(L_1, L_2)| \leq \pi/4 \\ \pi/4 & \text{otherwise} \end{cases}$$

   does this define a metric on X? Why or why not?

7. (optional) Show that if X is the set of continuous functions from the interval $[0, 1]$ to the real line $\mathbb{R}$, then setting $d(f, g) = \max\{|f(x) - g(x)| \mid x \in X\}$ defines a metric on X.

8. (optional) Let (X, d) be a metric space. Consider the following two definitions:

   Definition 1 A subset $S \subset X$ is called bounded if there exists some $x \in X$ and some real number $R > 0$ such that for all $s \in S$, $d(x, a) \leq R$.

   Definition 2 A subset $S \subset X$ is called boundid if there for every $x \in X$, there exists some real number $R > 0$ such that for all $s \in S$, $d(x, s) \leq R$.

   Show that a subset $S \subset X$ is bounded if and only if it is boundid.

9. (optional) Let X be a set and let $\mathcal{X} = \{d : X \times X \to \mathbb{R} | d \text{ is a metric}\}$ be the set of all metrics on X. We are going to try to put something like a metric on the space $\mathcal{X}$, to try to talk about how "far apart" metrics are from each other[1].

   For metrics d_1, d_2 define

$$M(d_1, d_2) = \inf\{r \in \mathbb{R} \mid d_1(x, y) \leq r d_2(x, y) \text{ for all } x, y \in X\}$$
$$m(d_1, d_2) = \sup\{r \in \mathbb{R} \mid r d_2(x, y) \leq d_1(x, y) \text{ for all } x, y \in X\}.$$

   That is, $M(d_1, d_2)$ and $m(d_1, d_2)$ are the smallest and largest real numbers respectively such that

$$m(d_1, d_2) d_2(x, y) \leq d_1(x, y) \leq M(d_1, d_2) d_2(x, y).$$

   Define $\delta(d_1, d_2) = \log \frac{M(d_1, d_2)}{m(d_1, d_2)}$.

   Show that this function δ satisfies the first, third and fouth properties of a metric (nonnegativity, symmetry, triangle inequality), but not the second (equality of indiscernibles).

[1] the issue of comparing different metrics on a space is a very natural one – often one has a set in question, be it phylogenetic trees, or some set of user preferences, and one would like to compare different ways of assigning distances to elements of this set!