Brooks' Theorem

The very natural general case

Brooks' Theorem Let G be a (simple) connected graph which is not complete and not an odd cycle.

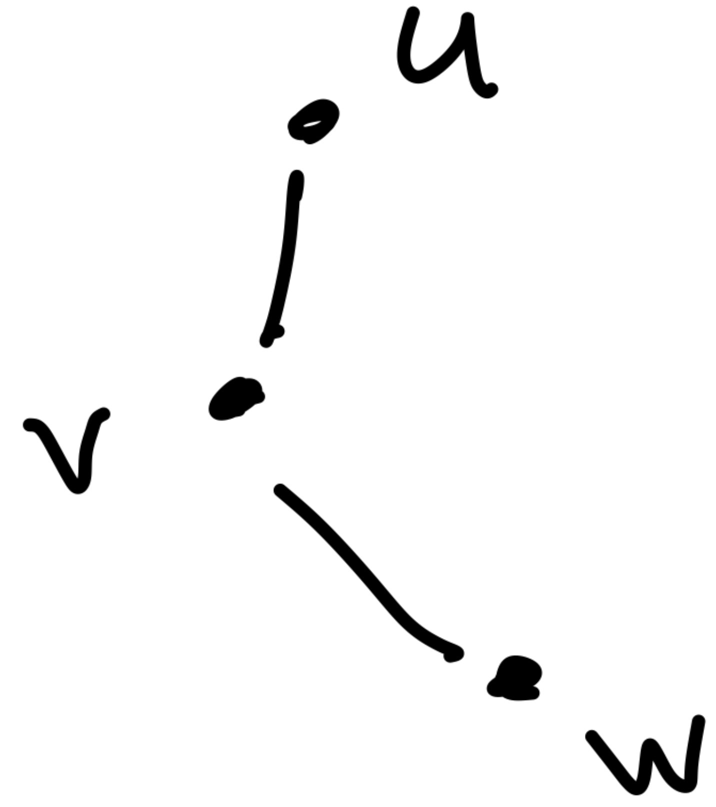
Then $\chi(G) \leq \Delta(G)$

Pf: Suppose we can find vertices u, v, w with

- $G - \{u, w\}$ connected
- $uv, vw \in E(G)$ but $uw \notin E(G)$.

Pf: Suppose we can find vertices u, v, w with

- $G - \{u, w\}$ connected
- $uv, vw \in E(G)$ but $uw \notin E(G)$.

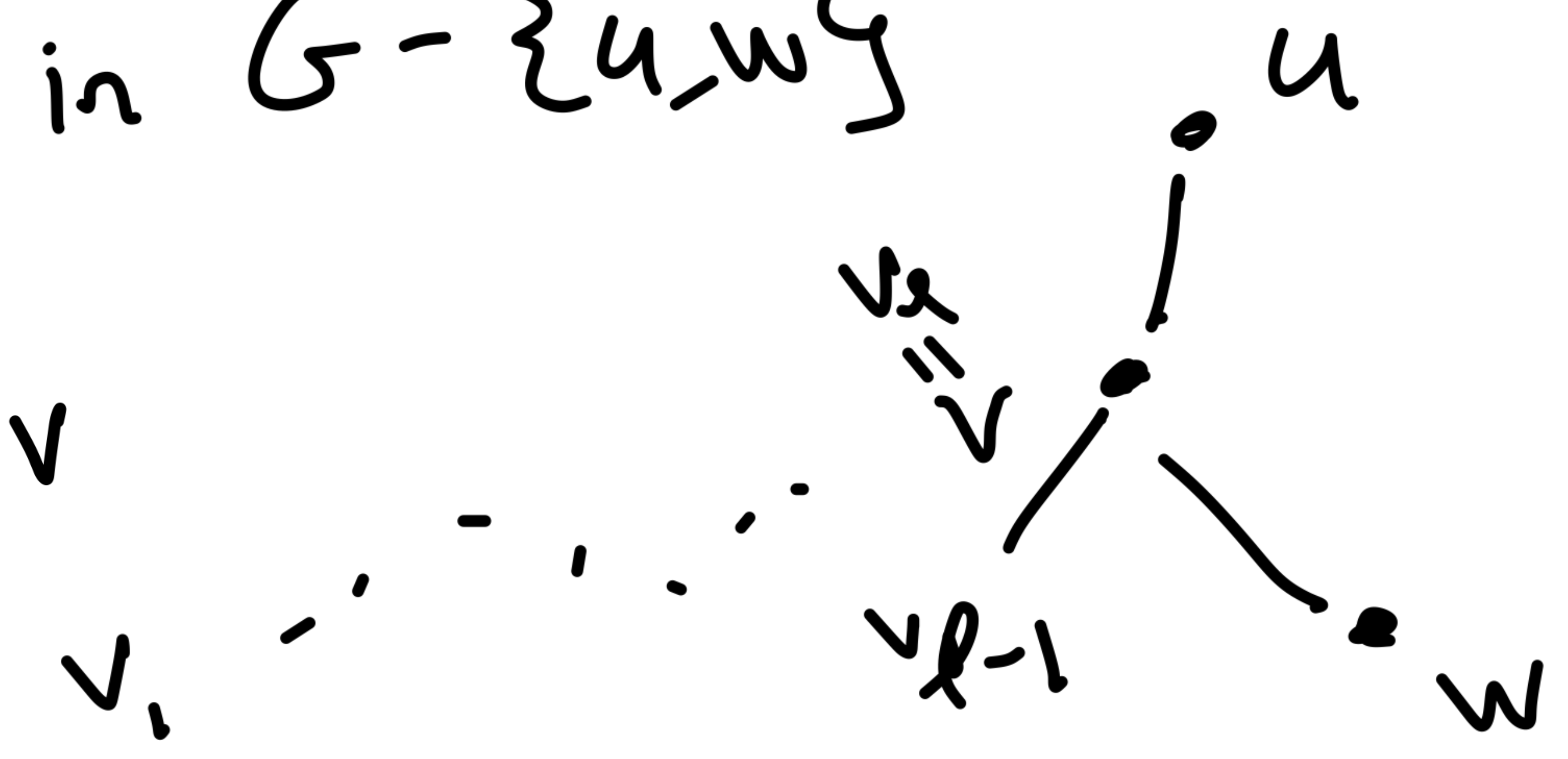


List vertices of $G - \{u, w\}$ as $v_1, v_2, \dots, v_\ell = v$ in decreasing order of distance from v in $G - \{u, w\}$

So: $v_{\ell-1}$ is adjacent to v, v_1 is as far as possible from v

$$d(v_i, v) \geq d(v_j, v) \text{ for } i < j$$

List vertices of $G - \{u,v,w\}$ as $v_1, v_2, \ldots, v_\ell = v$ in decreasing order of distance from v in $G - \{u,w\}$



$d(v_i, v) \geq d(v_j, v)$ for $i < j$ i.e. $v_i \neq v$

now: if $d(v_i, v) = k \geq 1$ then

v_i must be adjacent to some v_j w/ $d(v_j, v) = k-1$

$\Longrightarrow$ v_i adjacent to some v_j for $j > i$

<u>Algorithm:</u>

1. Color u & w w/ color 1.
2. Color $v_1, v_2, \ldots, v_\ell$ in order, using first available color from $\{1, \ldots, \Delta\}$

Algorithm: 1. Color u & w w/ color 1.
2. Color $v_1, v_2, \dots, v_\ell$ in order, using first available color from $\{1, \dots, \Delta\}$

why does this work?

Showed: v_i adjacent to some v_j for $j > i$ if $v_i \neq v$

So, when coloring vertex v_i, $i < \ell$, there is always v_j adjacent to v_i, with $j > i \Rightarrow v_j$ is uncolored

$\Rightarrow \exists$ at most $\Delta - 1$ neighbors of v_i w/ colors $\Rightarrow$ can choose some color in $\{1, \dots, \Delta\}$ for v_i.

Algorithm: 1. Color u & w w/ color 1.
2. Color $v_1, v_2, \ldots, v_\ell$ in order, using first available color from $\{1, \ldots, \Delta\}$

why does this work?

How about coloring $v_\ell = v$?

- u, w are adjacent to v and both have color 1,
- v has at most Δ neighbors
- neighbors have at most $\Delta - 1$ colors
- can choose a valid color for v! $\square$

What's wrong with this proof?

<u>Starting point:</u> Suppose we can find $u, v, w \in G$

with

- $G - \{u, w\}$ connected (used to construct v_i's – every vertex a finite distance from v in $G - \{u, w\}$)

- $uv, vw \in E(G)$, $uw \notin E(G)$

Lemma: Suppose G is connected & not complete. Then we can find $u, v, w \in G$ with $uv, vw \in E(G)$ and $uw \notin E(G)$.

Pf: Suppose this is not the case (argue by contradiction). Since G is not complete, we can find x, y such that $xy \notin E(G)$. Since G is connected we can find a walk

$v_1 v_2 \cdots v_m$ $\quad v_1 = x$, $v_m = y$. By assumption:

$v_1v_2, v_2v_3 \in E \Rightarrow v_1v_3 \in E$. $v_1v_3, v_3v_4 \in E \Rightarrow v_1v_4 \in E \ldots$

$\Rightarrow v_1 v_m \in E$ ↯ $\square$

Proof of Brooks' theorem

Let G be not complete, not an odd cycle, connected.

Suppose $\chi(G) > \Delta(G)$.

Remove edges if needed, get a new (critical) graph G' with $\chi(G') = \chi(G)$ $\Delta(G') \leq \Delta(G)$.

So $\chi(G') > \Delta(G')$. So WLOG can assume G is critical.

(and G' not complete, not odd cycle)

(else $\chi = 3$ $\Delta(G) \geq 3$)

As we have seen, critical $\Rightarrow$ block. So G is a block.

Suppose G has a 2-vertex cut.
We saw last time $\Rightarrow$ Brooks' theorem holds for G
so this can't be the case. So G is 3-connected.

Now, since G is not complete, can find u, v, w w/
$uv, vw \in E$, $uw \notin E$.

G 3-connected $\Rightarrow$ $G - \{u, w\}$ connected. Previous argument
$\Rightarrow$ Brooks holds.

So: If G is a counterexample to Brooks' thm

$\Rightarrow$ can find another graph which is a counterexample.
(G')

but can then show it isn't a counterexample.

$\Rightarrow$ there can be no counterexamples

$\Rightarrow$ Brooks' thm is true $\square$

:)