Towards Brooks' Theorem

The 2-connected case

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Brooks' Theorem Suppose G is a (simple) connected graph, G not complete and not an odd cycle. Then $\chi(G) \le \Delta(G)$.

Strategy (Lovász 1973 / Bondy & Murty 1976)

Argue by contradiction, careful examination of a "minimal criminal"

Def A graph G is called **k-critical** if $\chi(G) = k$ but $\chi(G-e) < k$ for all $e \in E(G)$.

(Standard def: A graph G is k-critical if $\chi(G) = k$, but $\chi(H) < k$ all $H \subsetneq G$)

Same if G is connected, which we'll always assume.

Recall: if $\chi(G)=k$ then $k \le \Delta(G)+1$

Brooks = $\boxed{k \le \Delta(G)}$

Theorem: If G is k-critical, then

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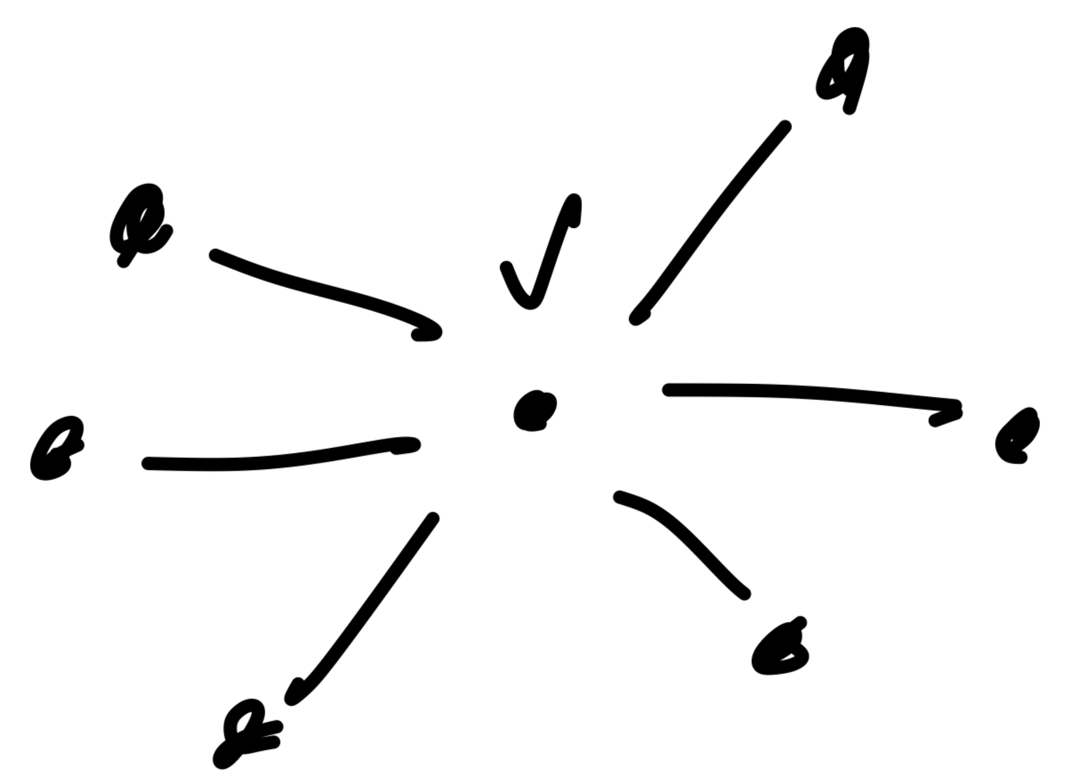
<u>Theorem</u>: If G is k-critical, then

$$k \leq \delta(G) + 1$$

<u>Pf</u>: Suppose $\deg(v) = \delta < k-1$. G k-critical

$\Rightarrow$ $G - v$ is $k-1$ colorable.



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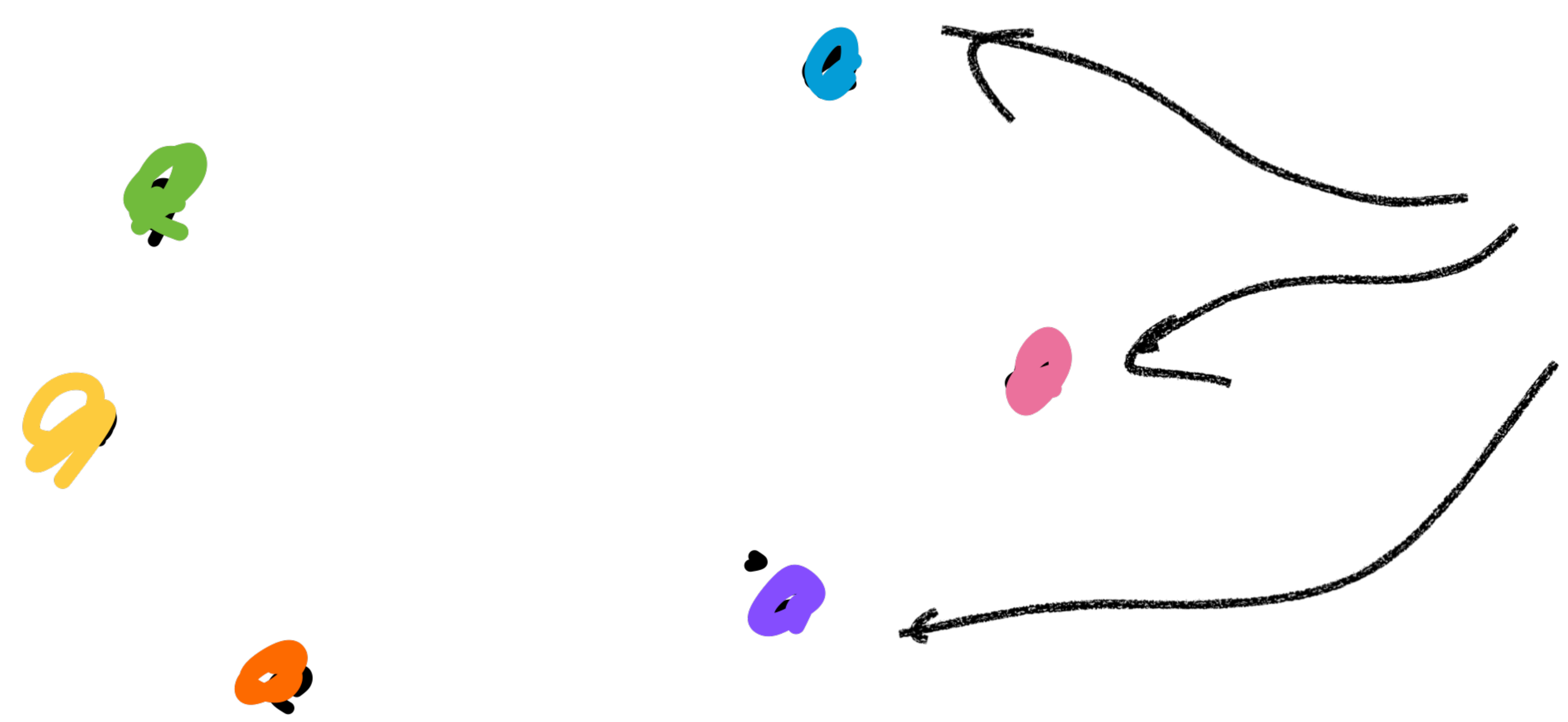

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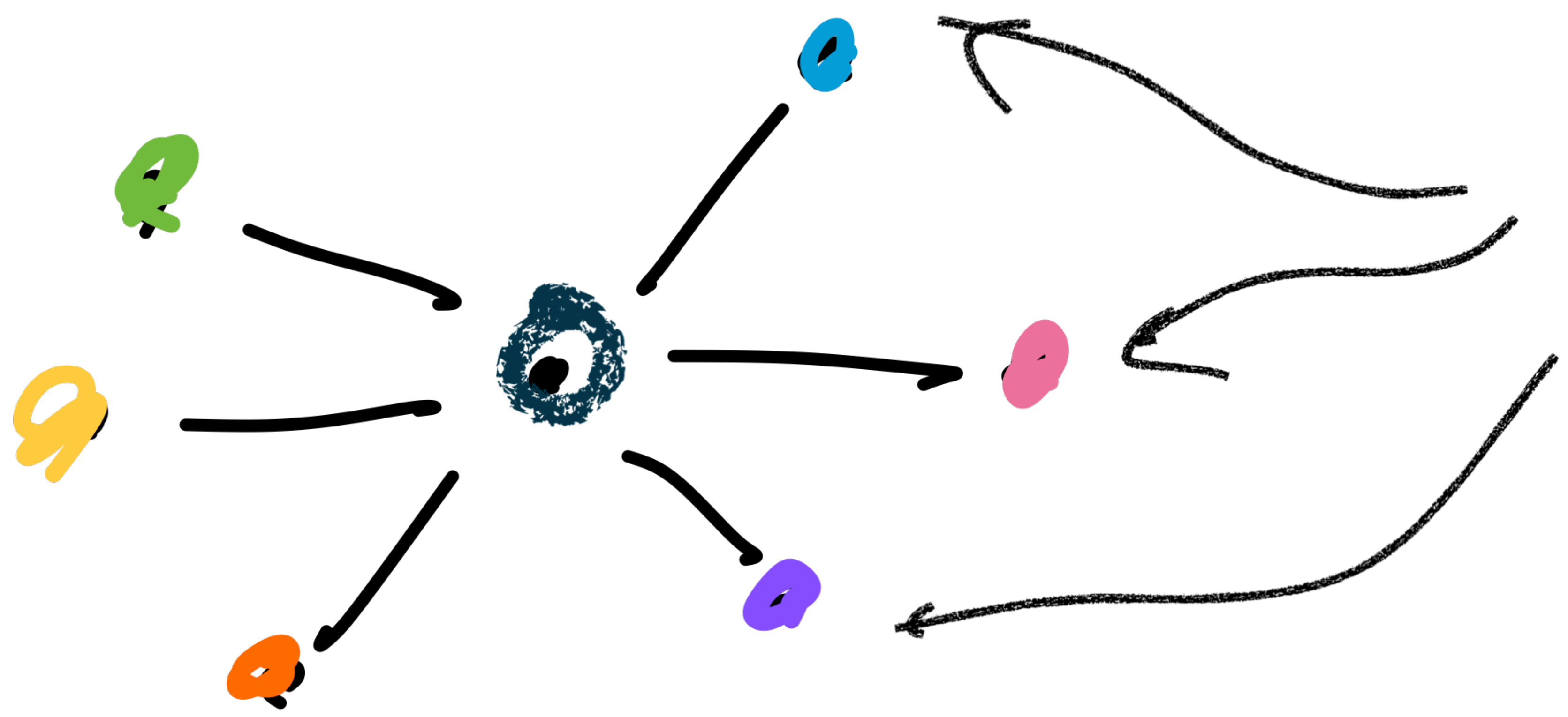

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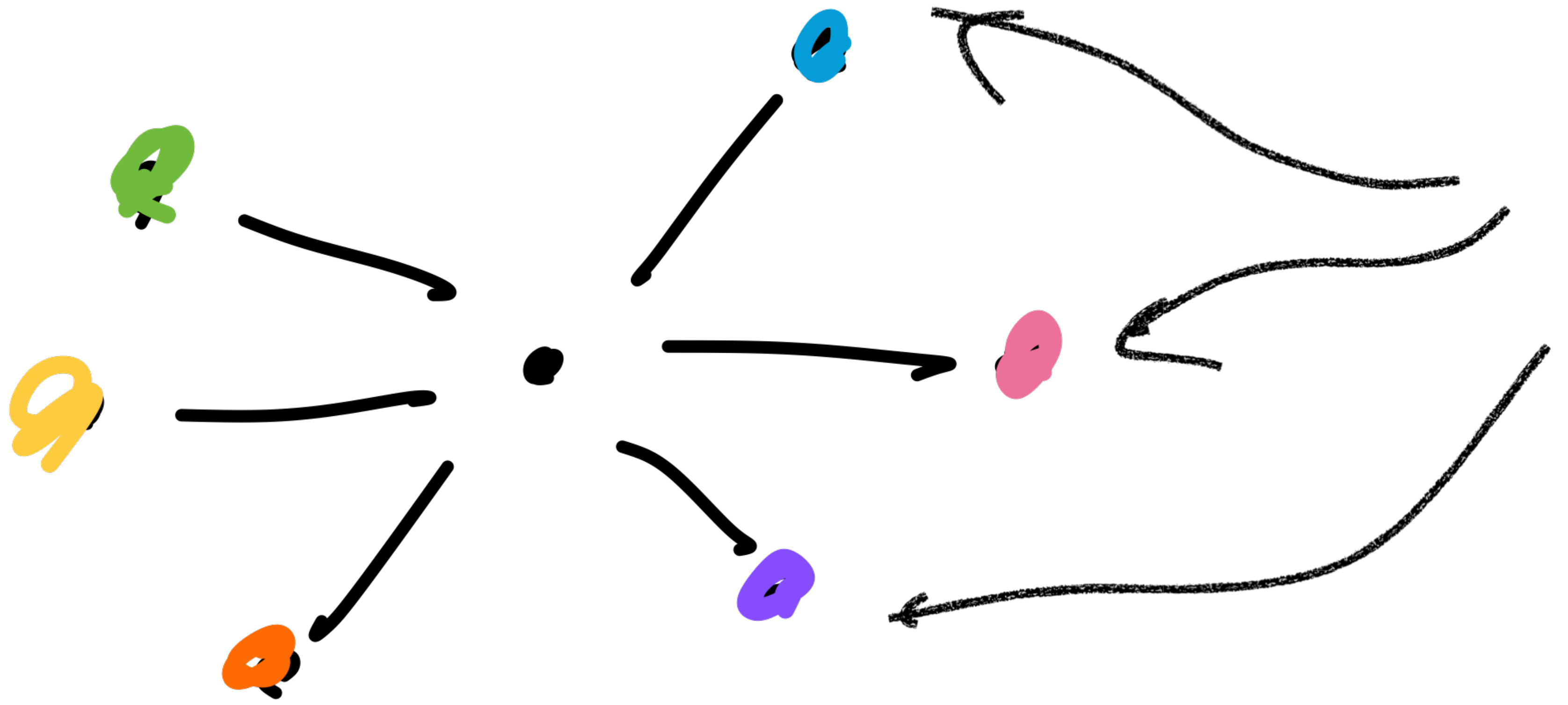
$\chi(G) \le k-1$ ↯ $\Leftarrow$



Before: no cut vertices

Question: what about 2-vertex cuts?

Answer: Very restrictive.

Prop: If u, v are a 2-vertex cut in a k-critical graph, then $\deg(u) + \deg(v) \geq 3k-5$

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Is this "good"?

Is this "useful"?

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Is this "good"?

previous says $\deg u \geq \delta(G) \geq k-1$

$\deg v \geq k-1$

so $\deg u + \deg v \geq 2k-2$!

Prop: If u, v are a 2-vertex cut in a k-critical graph, then $\deg(u) + \deg(v) \geq 3k-5$

Is this "good"?

so - gorgeously strong!

Prop: If u, v are a 2-vertex cut in a k-critical graph, then $\deg(u) + \deg(v) \geq 3k-5$

Is this "useful"?

If have such a cut $\Rightarrow$

$$2\Delta(G) \geq \deg u + \deg v \geq 3k-5 \geq 2k-1$$

if $k \geq 4$.

but $2\Delta(G)$ even $\Rightarrow$ $\Delta(G) \geq k$ ✓

Prop: If u, v are a 2-vertex cut in a k-critical graph, then $\deg(u) + \deg(v) \geq 3k-5$

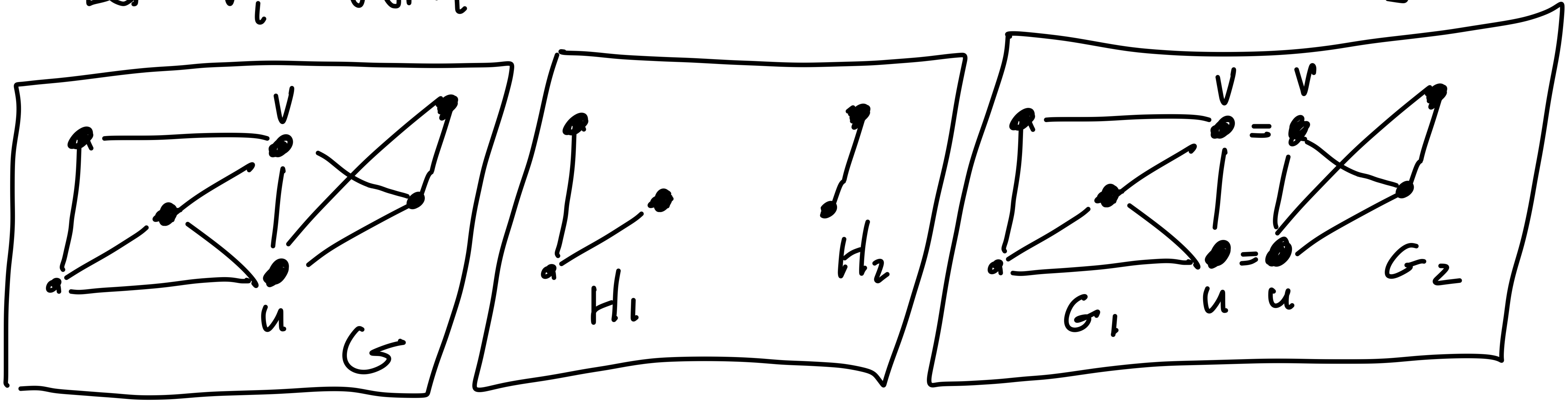
Is this "useful"?

consequence: if $\exists$ a 2-vertex cut, Brooks thm true for G.

Prop: If u, v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let $H_1, \dots, H_\ell$ be the components of $G - \{u,v\}$

Let $V_i = V(H_i)$ and set $G_i = G[V_i \cup \{u,v\}]$



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$G_i \subsetneq G \implies G_i$ is $k-1$ colorable

Suppose we can color all G_i's w/ $k-1$ colors giving

u & v same color $\implies$ can color G w/ $k-1$ colors $\lightning$

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let $H_1, \dots, H_\ell$ be the components of $G-\{u,v\}$

Let $V_i = V(H_i)$ and set $G_i = G[V_i \cup \{u,v\}]$

$G_i \subsetneq G \Rightarrow G_i$ is $k-1$ colorable

Suppose we can color all G_i's w/ $k-1$ colors giving u & v different colors $\Rightarrow$ can color G w/ $k-1$ colors

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let $H_1, \ldots, H_\ell$ be the components of $G - \{u,v\}$

Let $V_i = V(H_i)$ and set $G_i = G[V_i \cup \{u,v\}]$

WLOG, every $k-1$ coloring of G_1 gives u,v diff colors

& every $k-1$ coloring of G_2 gives u,v same colors

$\Rightarrow G_1 \cup G_2$ is k-critical $\Rightarrow G = G_1 \cup G_2$!

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let $H_1, \ldots, H_\ell$ be the components of $G - \{u,v\}$

Let $V_i = V(H_i)$ and set $G_i = G[V_i \cup \{u,v\}]$

$G_i \subsetneq G \Rightarrow G_i$ is $k-1$ colorable

so not adjacent!

$\ell = 2$, every colorg of G_1 w/ $k-1$ colors gives u,v diff colors & every colorg of G_2 w/ $k-1$ gives u,v same color!

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let H_1, H_2 be the components of $G - \{u,v\}$

$H_1/_{u,v}$ is not $k-1$ colorable

$H_2 + uv$ is not $k-1$ colorable

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let H_1, H_2 be the components of $G-\{u,v\}$

$H_1/_{u,v}$ is not $k-1$ colorable
but it is k-colorable :

coloring of $H_1/_{u,v}$

$\Uparrow$

$G-\{u,v\}$ is $k-1$ colorable $\Rightarrow$ G can be k colored by assigning new color to both u & v

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let H_1, H_2 be the components of $G-\{u,v\}$

$H_1/_{u,v}$ is <u>k-critical</u>

Let $e \in E(H_1) = E(H_1/_{u,v})$ then $G-e$ is $k-1$ colorable. Restricting to G_2: u,v get same color in this coloring $\Rightarrow$ get a coloring of $H_1/_{u,v}$

Prop: If u, v are a 2-vertex cut in a k-critical graph, then $\deg(u) + \deg(v) \geq 3k-5$

Pf: Let H_1, H_2 be the components of $G - \{u,v\}$

$H_2 + uv$ is <u>k-critical</u>

k-colorable: $G - u$ is $k-1$ colorable $\implies$

can add u w/ kth color, get a k-coloring of

G w/ u, v diff colors ✓

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let H_1, H_2 be the components of $G-\{u,v\}$

H_2+uv is k-critical k-colorable ✓

$(H_2+uv)-uv = H_2$ is $k-1$ colorable.

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let H_1, H_2 be the components of $G-\{u,v\}$

H_2+uv is $\underline{k\text{-critical}}$ $\quad$ k-colorable ✓

If $e \in E(H_2)$, $G-e$ is $k-1$ colorable, restrict coloring to $G_1 \Rightarrow u,v$ diff. colors $\Rightarrow$ get a coloring of H_2 ✓

Prop: If u,v are a 2-vertex cut in a k-critical graph, then $\deg(u)+\deg(v) \geq 3k-5$

Pf: Let H_1, H_2 be the components of $G - \{u,v\}$

$G_1/u,v$ & $G_2 + uv$ are k-crit. $\Longrightarrow \delta(G_i) \geq k-1$

so $\deg_{G_2+uv}(u), \deg_{G_2+uv}(v) \geq k-1 \rightsquigarrow 2\cdot(k-2)$ edges in G

$\deg_{G_1/u,v}(uv) \geq k-1 \rightsquigarrow k-1$ edges in G

$\Longrightarrow 2k-4+k-1 = 3k-5$ edges!