Notes on derived categories and motives

Daniel Krashen

Table of Contents

Table of Contents

Motives $\leftrightarrow$ Derived categories

moral similarity

both sit in between geometric objects and thier cohomology

Motives $\leftrightarrow$ Derived categories

moral similarity

both sit in between geometric objects and thier cohomology

differences

Motives $\leftrightarrow$ Derived categories

moral similarity

both sit in between geometric objects and thier cohomology

differences

- different kinds of things: object in Abelian category vs triangulated category

Motives $\leftrightarrow$ Derived categories

moral similarity

both sit in between geometric objects and thier cohomology

differences

- different kinds of things: object in Abelian category vs triangulated category
- designed to handle different kinds of decompositions: spaces vs coefficients

Comparison: K-theory and Chow groups

analogy

The derived category is to motives as
K-theory is to Chow groups

These are related via the Chern character / Riemann-Roch

Comparison: K-theory and Chow groups

analogy

The derived category is to motives as
K-theory is to Chow groups

These are related via the Chern character / Riemann-Roch

enrichments

The derived category carries richer information than K-theory, and
Motives carry richer information than Chow groups.

Comparison: K-theory and Chow groups

analogy

The derived category is to motives as
K-theory is to Chow groups

These are related via the Chern character / Riemann-Roch

enrichments

The derived category carries richer information than K-theory, and
Motives carry richer information than Chow groups.

Question

Is it possible that these carry very similar information at the end?

Table of Contents

Table of Contents

cochain complexes

Definition

For an Abelian category $\mathscr{A}$, let $coCh^*(\mathscr{A})$ (where $*$ is either "empty" or is one of the symbols $+, -, b$), be the category whose objects are sequences of objects and morphisms of $\mathscr{A}$ of the form:

$$A^\bullet = \cdots \xrightarrow{d_{i-1}} A^i \xrightarrow{d_i} A^{i+1} \xrightarrow{d_{i+1}} \cdots$$

where $A^n = 0$ if $n >> 0$ in case $* = +$, or if $n << 0$ in case $* = -$, or if $|n| >> 0$ in case $* = b$, and such that $d^{i+1}d^i = 0$ for all i.

Morphisms $f^\bullet : A^\bullet \to B^\bullet$ defined to be collections of morphisms $f^i : A^i \to B^i$ such that we have commutative diagrams:

$$\begin{array}{ccc} A^i & \longrightarrow & A^{i+1} \\ \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} \\ B^i & \longrightarrow & B^{i+1} \end{array}$$

quasi-isomorphisms

Definition

$\mathcal{H}^n(A) = \frac{\ker\left(d:A^n \to A^{n+1}\right)}{\operatorname{im}\left(d:A^{n-1} \to A^n\right)}$.

$f^\bullet : A^\bullet \to B^\bullet$ induces $\mathcal{H}^n(f^\bullet) : \mathcal{H}^n(A^\bullet) \to \mathcal{H}^n(B^\bullet)$.

quasi-isomorphisms

Definition

$\mathcal{H}^n(A) = \frac{\ker\left(d:A^n \to A^{n+1}\right)}{\operatorname{im}\left(d:A^{n-1} \to A^n\right)}$.

$f^\bullet : A^\bullet \to B^\bullet$ induces $\mathcal{H}^n(f^\bullet) : \mathcal{H}^n(A^\bullet) \to \mathcal{H}^n(B^\bullet)$.

Definition

$f^\bullet : A^\bullet \to B^\bullet$ is a quasi-isomorphism if $\mathcal{H}(f^\bullet)$ is an isomorphism for all n.

localization of a category

Theorem

Let $\mathscr{B}$ be an arbitrary category and S an arbitrary class of morphisms of $\mathscr{B}$. Then there exists a category $\mathscr{B}[S^{-1}]$ and a functor $Q : \mathscr{B} \to \mathscr{B}[S^{-1}]$ with the following universal property:

localization of a category

Theorem

Let $\mathscr{B}$ be an arbitrary category and S an arbitrary class of morphisms of $\mathscr{B}$. Then there exists a category $\mathscr{B}[S^{-1}]$ and a functor $Q : \mathscr{B} \to \mathscr{B}[S^{-1}]$ with the following universal property:

- *$Q(f)$ is an isomorphism for every $f \in S$,*

localization of a category

Theorem

Let $\mathscr{B}$ be an arbitrary category and S an arbitrary class of morphisms of $\mathscr{B}$. Then there exists a category $\mathscr{B}[S^{-1}]$ and a functor $Q : \mathscr{B} \to \mathscr{B}[S^{-1}]$ with the following universal property:

- *$Q(f)$ is an isomorphism for every $f \in S$,*
- *given any functor $F : \mathscr{B} \to \mathscr{D}$ such that $F(f)$ is an isomorphism for every $f \in S$, there exists a unique functor $G : B[S^{-1}] \to D$ such that $F = G \circ Q$.*

localization of a category

Theorem

Let $\mathscr{B}$ be an arbitrary category and S an arbitrary class of morphisms of $\mathscr{B}$. Then there exists a category $\mathscr{B}[S^{-1}]$ and a functor $Q : \mathscr{B} \to \mathscr{B}[S^{-1}]$ with the following universal property:

- *$Q(f)$ is an isomorphism for every $f \in S$,*
- *given any functor $F : \mathscr{B} \to \mathscr{D}$ such that $F(f)$ is an isomorphism for every $f \in S$, there exists a unique functor $G : B[S^{-1}] \to D$ such that $F = G \circ Q$.*

Definition

For an Abelian category $\mathscr{A}$, let $QI^*(\mathscr{A})$ to be the collection of quasi-isomorphisms in $coCh^*(\mathscr{A})$. We define:

$$D^*(\mathscr{A}) = coCh^*(\mathscr{A})[(QI^*(\mathscr{A})^{-1}].$$

$D^*(X)$

Definition

Let X be a scheme. We define $D^*(X)$ to be the derived category $D^*(Coh(X))$ where $Coh(X)$ is the Abelian category of coherent sheaves on X.

$D^*(X)$

Definition

Let X be a scheme. We define $D^*(X)$ to be the derived category $D^*(Coh(X))$ where $Coh(X)$ is the Abelian category of coherent sheaves on X.

Problems

$D^*(X)$

Definition

Let X be a scheme. We define $D^*(X)$ to be the derived category $D^*(Coh(X))$ where $Coh(X)$ is the Abelian category of coherent sheaves on X.

Problems

- triangulated structure of $D^*(X)$ not apparent,

$D^*(X)$

Definition

Let X be a scheme. We define $D^*(X)$ to be the derived category $D^*(Coh(X))$ where $Coh(X)$ is the Abelian category of coherent sheaves on X.

Problems

- triangulated structure of $D^*(X)$ not apparent,
- hom sets difficult to compute and compose.

$D^*(X)$

Definition

Let X be a scheme. We define $D^*(X)$ to be the derived category $D^*(Coh(X))$ where $Coh(X)$ is the Abelian category of coherent sheaves on X.

Problems

- triangulated structure of $D^*(X)$ not apparent,
- hom sets difficult to compute and compose.

Solutions

$D^*(X)$

Definition

Let X be a scheme. We define $D^*(X)$ to be the derived category $D^*(Coh(X))$ where $Coh(X)$ is the Abelian category of coherent sheaves on X.

Problems

- triangulated structure of $D^*(X)$ not apparent,
- hom sets difficult to compute and compose.

Solutions

- alternate, more concrete construction,

$D^*(X)$

Definition

Let X be a scheme. We define $D^*(X)$ to be the derived category $D^*(Coh(X))$ where $Coh(X)$ is the Abelian category of coherent sheaves on X.

Problems

- triangulated structure of $D^*(X)$ not apparent,
- hom sets difficult to compute and compose.

Solutions

- alternate, more concrete construction,
- comparison of derived categories of related Abelian categories.

Table of Contents

Notational preliminaries

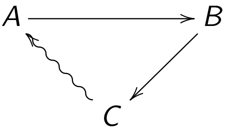
Let $\mathscr{T}$ be an additive category, $T : \mathscr{T} \to \mathscr{T}$ an additive equivalence (autequivalence).

Notation

We will write $A \overset{f}{\rightsquigarrow} B$ to mean f is a morphism from A to TB.

Warning: this is not a standard notation!

or equivalently



Morphisms of triangular diagrams are collections of morphisms making commutative diagrams.

Notational preliminaries

Let $\mathscr{T}$ be an additive category, $T : \mathscr{T} \to \mathscr{T}$ an additive equivalence (autequivalence).

Notation

We will write $A \overset{f}{\rightsquigarrow} B$ to mean f is a morphism from A to TB.

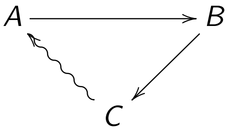
Warning: this is not a standard notation!

Notation

A **triangular diagram** is a collection of objects and morphisms of the form

$$A \to B \to C \to TA.$$

or equivalently



Morphisms of triangular diagrams are collections of morphisms making commutative diagrams.

Definition overview

Definition

A **triangulated category** *is an additive category* $\mathscr{T}$ *with an autoequivalence* T*, and a class of triagular diagrams* Δ*, called distinguished triangles, which satisfy*

Definition overview

Definition

A **triangulated category** *is an additive category* $\mathscr{T}$ *with an autoequivalence* T*, and a class of triagular diagrams* Δ*, called distinguished triangles, which satisfy*

- *Axiom TR1*

Definition overview

Definition

A **triangulated category** *is an additive category $\mathscr{T}$ with an autoequivalence T, and a class of triagular diagrams Δ, called distinguished triangles, which satisfy*

- *Axiom TR1*
- *Axiom TR2*

Definition overview

Definition

A **triangulated category** *is an additive category* $\mathscr{T}$ *with an autoequivalence* T*, and a class of triagular diagrams* Δ*, called distinguished triangles, which satisfy*

- *Axiom TR1*
- *Axiom TR2*
- *Axiom TR3*

Definition overview

Definition

A **triangulated category** *is an additive category* $\mathscr{T}$ *with an autoequivalence* T*, and a class of triagular diagrams* Δ*, called distinguished triangles, which satisfy*

- *Axiom TR1*
- *Axiom TR2*
- *Axiom TR3*
- *Axiom TR4*

Axiom TR1: some triangles you must have

Axiom (TR1)

We say that $\mathscr{T}, T, \Delta$ *satisfies TR1 if:*

Axiom TR1: some triangles you must have

Axiom (TR1)

We say that $\mathscr{T}, T, \Delta$ *satisfies TR1 if:*

i. *For any* A*, the triangular diagram* $A \xrightarrow{id} A \to 0 \to TA$ *is in* Δ*,*

Axiom TR1: some triangles you must have

Axiom (TR1)

We say that $\mathscr{T}, T, \Delta$ satisfies TR1 if:

i. *For any A, the triangular diagram $A \xrightarrow{id} A \to 0 \to TA$ is in Δ,*

ii. *Any triangular diagram isomorphic to one in Δ is also in Δ,*

Axiom TR1: some triangles you must have

Axiom (TR1)

We say that $\mathscr{T}, T, \Delta$ *satisfies TR1 if:*

i. *For any* A, *the triangular diagram* $A \xrightarrow{id} A \to 0 \to TA$ *is in* Δ,
ii. *Any triangular diagram isomorphic to one in* Δ *is also in* Δ,
iii. *Every morphism* $A \to B$ *can be completed to a triangular diagram* $A \to B \to C \to TA$ *which is in* Δ.

Axiom TR2: rotation

Axiom (TR2)

We say that $\mathscr{T}$ *satisfies axiom TR2 if whenever* $A \xrightarrow{f} B \to C \to TA$ *is in* Δ*, the diagram* $B \to C \to TA \xrightarrow{-Tf} TB$ *is also in* Δ*.*

Axiom TR3: existence of morphisms between triangles

Axiom (TR3)

We say that $\mathscr{T}$ satisfies axiom TR3 if the following holds.
Given $A \to B \to C \to TA$ and $A' \to B' \to C' \to TA'$ in Δ, and a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ A' & \longrightarrow & B', \end{array}$$

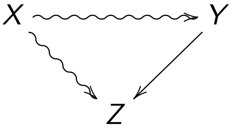
we may find a morphism $c : C \to C'$ giving rise to a morphism of triangular diagrams

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle Ta} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

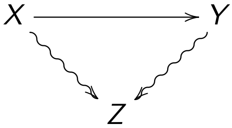
temporary notational convention

Definition

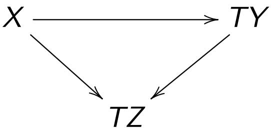
We will say that diagrams such as



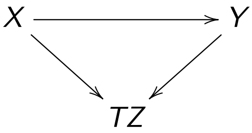
or



commute if the corresponding diagrams



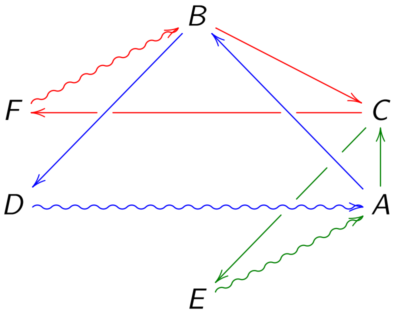
or



commute.

Axiom T4: compatibility of morphisms between triangles

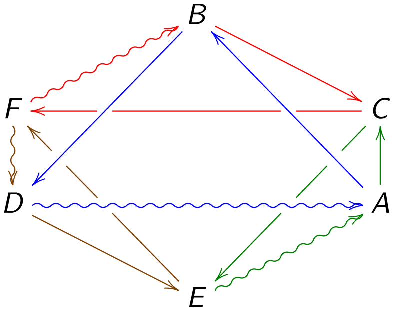
Suppose we are given a diagram of the form



Where the triangle on the upper right is commutative, and the three monochromatic triangular diagrams are in Δ. Then we may find morphisms $D \to E \to F \to TD$ such that...

Axiom T4: compatibility of morphisms between triangles

in the diagram



every monochromatic triangular subdiagram is in Δ and every tricolored face is commutative.

Note – TR3 ensures the existence of maps $E \to F$ and $D \to E$ which make the diagram commutative (ignoring the remaining brown side). TR4 ensures that one can make the entire diagram compatible.

Table of Contents

The Homotopy Category

Passing to the derived category factors through the homotopy category:

$$coCh^*(\mathscr{A}) \to K^*(\mathscr{A}) \to D^*(\mathscr{A})$$

Advantages of $K^*(\mathscr{A})$:

- it is clearly additive,

The Homotopy Category

Passing to the derived category factors through the homotopy category:

$$coCh^*(\mathscr{A}) \to K^*(\mathscr{A}) \to D^*(\mathscr{A})$$

Advantages of $K^*(\mathscr{A})$:

- it is clearly additive,
- in fact: it is a triangulated category,

The Homotopy Category

Passing to the derived category factors through the homotopy category:

$$coCh^*(\mathscr{A}) \to K^*(\mathscr{A}) \to D^*(\mathscr{A})$$

Advantages of $K^*(\mathscr{A})$:

- it is clearly additive,
- in fact: it is a triangulated category,
- it's morphisms are easy to describe and compose,

The Homotopy Category

Passing to the derived category factors through the homotopy category:

$$coCh^*(\mathscr{A}) \to K^*(\mathscr{A}) \to D^*(\mathscr{A})$$

Advantages of $K^*(\mathscr{A})$:

- it is clearly additive,
- in fact: it is a triangulated category,
- it's morphisms are easy to describe and compose,
- we can construct $D^*(\mathscr{A})$ from it by a much simpler localization process.

Chain Homotopies

Let $\mathscr{A}$ be an **additive** category, and let A, B be cochain complexes in $\mathscr{A}$.

Definition

Given $f, g : A \to B$, *a* **(cochain) homotopy** $h : f \to g$ *is a collection of maps* $h^i : A^i \to B^{i-1}$ *such that* $g - f = dh + hd$.

If such a homotopy exists, we say that f and g are **homotopic**. If a morphism is homotopic to the 0 map, we say it is **null-homotopic**.

Chain Homotopies

Let $\mathscr{A}$ be an **additive** category, and let A, B be cochain complexes in $\mathscr{A}$.

Definition

Given $f, g : A \to B$, a **(cochain) homotopy** *$h : f \to g$ is a collection of maps $h^i : A^i \to B^{i-1}$ such that $g - f = dh + hd$.*

If such a homotopy exists, we say that f and g are **homotopic**. If a morphism is homotopic to the 0 map, we say it is **null-homotopic**.

Definition

$K^*(\mathscr{A})$ is the category with

Chain Homotopies

Let $\mathscr{A}$ be an **additive** category, and let A, B be cochain complexes in $\mathscr{A}$.

Definition

Given $f, g : A \to B$, a **(cochain) homotopy** *$h : f \to g$ is a collection of maps $h^i : A^i \to B^{i-1}$ such that $g - f = dh + hd$.*

If such a homotopy exists, we say that f and g are **homotopic**. If a morphism is homotopic to the 0 map, we say it is **null-homotopic**.

Definition

$K^*(\mathscr{A})$ is the category with

- *same objects as $coCh^*(\mathscr{A})$,*

Chain Homotopies

Let $\mathscr{A}$ be an **additive** category, and let A, B be cochain complexes in $\mathscr{A}$.

Definition

Given $f, g : A \to B$, a **(cochain) homotopy** *$h : f \to g$ is a collection of maps $h^i : A^i \to B^{i-1}$ such that $g - f = dh + hd$.*

If such a homotopy exists, we say that f and g are **homotopic**. If a morphism is homotopic to the 0 map, we say it is **null-homotopic**.

Definition

$K^*(\mathscr{A})$ is the category with

- *same objects as $coCh^*(\mathscr{A})$,*
- *morphisms are homotopy classes of morphisms in $coCh^*(\mathscr{A})$.*

Chain Homotopies

Let $\mathscr{A}$ be an **additive** category, and let A, B be cochain complexes in $\mathscr{A}$.

Definition

Given $f, g : A \to B$, a **(cochain) homotopy** *$h : f \to g$ is a collection of maps $h^i : A^i \to B^{i-1}$ such that $g - f = dh + hd$.*

If such a homotopy exists, we say that f and g are **homotopic**. If a morphism is homotopic to the 0 map, we say it is **null-homotopic**.

Definition

$K^*(\mathscr{A})$ is the category with

- *same objects as $coCh^*(\mathscr{A})$,*
- *morphisms are homotopy classes of morphisms in $coCh^*(\mathscr{A})$.*

It turns out that $K^*(\mathscr{A})$ is triangulated!

The triangulated structure of $K^*(\mathscr{A})$

Features are visible already in $coCh^*(\mathscr{A})$:

- T is the "shift" $A \mapsto A[1]$,

The triangulated structure of $K^*(\mathscr{A})$

Features are visible already in $coCh^*(\mathscr{A})$:

- T is the "shift" $A \mapsto A[1]$,
- distinguished triangles are induced by term-wise split exact sequences of complexes.

The triangulated structure of $K^*(\mathscr{A})$

Features are visible already in $coCh^*(\mathscr{A})$:

- T is the "shift" $A \mapsto A[1]$,
- distinguished triangles are induced by term-wise split exact sequences of complexes.

For $0 \to A \to B \to C \to 0$ termwise split, choose $s : C \to B$ a termwise splitting (not a cochain map!).

The triangulated structure of $K^*(\mathscr{A})$

Features are visible already in $coCh^*(\mathscr{A})$:

- T is the "shift" $A \mapsto A[1]$,
- distinguished triangles are induced by term-wise split exact sequences of complexes.

For $0 \to A \to B \to C \to 0$ termwise split, choose $s : C \to B$ a termwise splitting (not a cochain map!).
Define $C \to A[1]$ by $ds - sd$.

The triangulated structure of $K^*(\mathscr{A})$

Features are visible already in $coCh^*(\mathscr{A})$:

- T is the "shift" $A \mapsto A[1]$,
- distinguished triangles are induced by term-wise split exact sequences of complexes.

For $0 \to A \to B \to C \to 0$ termwise split, choose $s : C \to B$ a termwise splitting (not a cochain map!).
Define $C \to A[1]$ by $ds - sd$.

Definition

A triangular diagram is distinguished if it is of the form

$$A \to B \to C \to A[1]$$

for $0 \to A \to B \to C \to 0$ *termwise split exact, and* $C \to A[1]$ *constructed as above.*

The triangulated structure of $K^*(\mathscr{A})$

Let $\mathscr{A}$ be an additive category and consider $K(\mathscr{A})$.

The triangulated structure of $K^*(\mathscr{A})$

Let $\mathscr{A}$ be an additive category and consider $K(\mathscr{A})$.
Let T be the autoequivalence induced by the shift, and let Δ be the class of triangular diagrams which are images of distinguished triangles in $coCh(\mathscr{A})$.

The triangulated structure of $K^*(\mathscr{A})$

Let $\mathscr{A}$ be an additive category and consider $K(\mathscr{A})$.
Let T be the autoequivalence induced by the shift, and let Δ be the class of triangular diagrams which are images of distinguished triangles in $coCh(\mathscr{A})$.
These give $\mathscr{A}$ the structure of a triangulated category.

How to get TR3

To extend $f : A \to B$ to a distinguished triangle, one uses the cone and cylinder constructions (of cochain complexes):

The triangulated structure of $K^*(\mathscr{A})$

Let $\mathscr{A}$ be an additive category and consider $K(\mathscr{A})$.
Let T be the autoequivalence induced by the shift, and let Δ be the class of triangular diagrams which are images of distinguished triangles in $coCh(\mathscr{A})$.
These give $\mathscr{A}$ the structure of a triangulated category.

How to get TR3

To extend $f : A \to B$ to a distinguished triangle, one uses the cone and cylinder constructions (of cochain complexes):

- isomorphism in the homotopy category $B \cong Cyl(f)$

The triangulated structure of $K^*(\mathscr{A})$

Let $\mathscr{A}$ be an additive category and consider $K(\mathscr{A})$.
Let T be the autoequivalence induced by the shift, and let Δ be the class of triangular diagrams which are images of distinguished triangles in $coCh(\mathscr{A})$.
These give $\mathscr{A}$ the structure of a triangulated category.

How to get TR3

To extend $f : A \to B$ to a distinguished triangle, one uses the cone and cylinder constructions (of cochain complexes):

- isomorphism in the homotopy category $B \cong Cyl(f)$
- induced map $A \to Cyl(f)$ fits into a termwise split exact sequence:

$$A \to Cyl(f) \to C(f)$$

The triangulated structure of $K^*(\mathscr{A})$

Let $\mathscr{A}$ be an additive category and consider $K(\mathscr{A})$.
Let T be the autoequivalence induced by the shift, and let Δ be the class of triangular diagrams which are images of distinguished triangles in $coCh(\mathscr{A})$.
These give $\mathscr{A}$ the structure of a triangulated category.

How to get TR3

To extend $f : A \to B$ to a distinguished triangle, one uses the cone and cylinder constructions (of cochain complexes):

- isomorphism in the homotopy category $B \cong Cyl(f)$
- induced map $A \to Cyl(f)$ fits into a termwise split exact sequence:

$$A \to Cyl(f) \to C(f)$$

- get a distinguished triangle.

$K^*(\mathscr{A})$ is still not good enough

For $\mathscr{A}$ Abelian, a short exact sequence $0 \to A \to B \to C \to 0$ of complexes need not correspond to a distinguished triangle in $D^*(\mathscr{A})$.
The derived category will fix this:

Proposition

Let $0 \to A \xrightarrow{f} B \to C \to 0$ be a short exact sequences of complexes. Then we have a morphism of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & Cyl(f) & \longrightarrow & C(f) & \longrightarrow & 0
\end{array}$$

with the vertical maps all quasi-isomorphisms.

$K^*(\mathscr{A})$ is still not good enough

For $\mathscr{A}$ Abelian, a short exact sequence $0 \to A \to B \to C \to 0$ of complexes need not correspond to a distinguished triangle in $D^*(\mathscr{A})$.
The derived category will fix this:

Proposition

Let $0 \to A \xrightarrow{f} B \to C \to 0$ be a short exact sequences of complexes. Then we have a morphism of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & Cyl(f) & \longrightarrow & C(f) & \longrightarrow & 0
\end{array}$$

with the vertical maps all quasi-isomorphisms.

Every short exact sequence becomes (part of) a distinguished triangle in the derived category.

Table of Contents

Table of Contents

Table of Contents

Table of Contents

References