Lecture 9: arclength

Thursday, September 4, 2014 12:31 PM

How to compute arclength. (6.3)

Parametric curves:

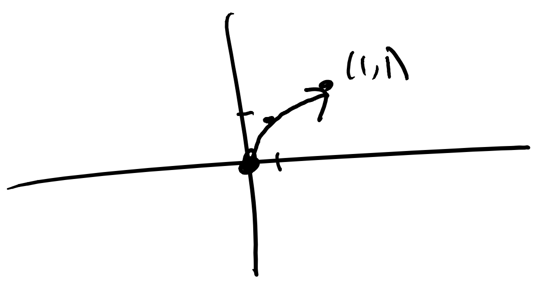
curve given by $x(t), y(t)$ point in plane, moving as a function of t.

ex: $\boxed{x(t) = t^2 \quad y(t) = t}$ $0 \le t \le 1$ $\rightsquigarrow x = y^2$

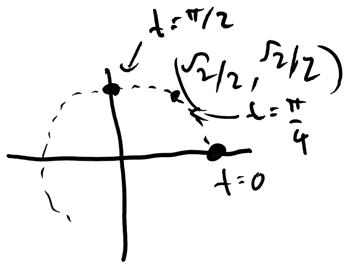
$t = 0 \rightsquigarrow (0^2, 0) = (0,0)$

$t = 1 \rightsquigarrow (1^2, 1)$

$t = \frac{1}{2} \rightsquigarrow \left(\frac{1}{4}, \frac{1}{2}\right)$



ex: $x(t) = \cos t \quad y(t) = \sin t$



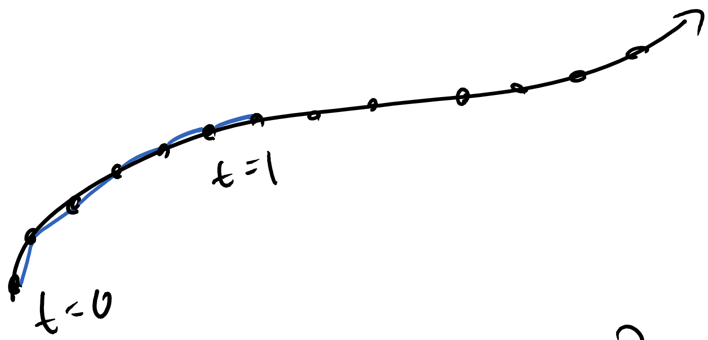
$t = 0 \rightsquigarrow (\cos 0, \sin 0) = (1, 0)$

$t = \frac{\pi}{4} \rightsquigarrow \left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

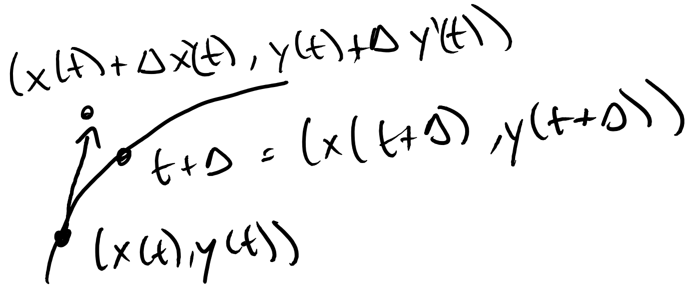
$t = \frac{\pi}{2} \rightsquigarrow \left(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right) = (0, 1)$

pt on unit circle at $\measuredangle\, t$ $\leftrightarrow (\cos t, \sin t)$

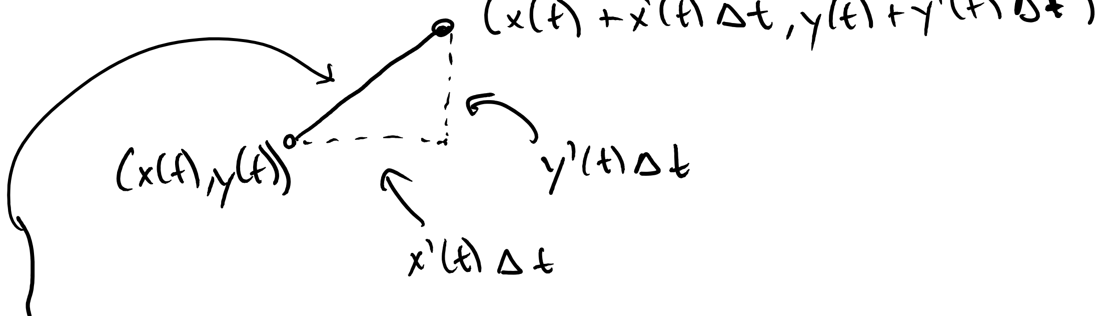
Basic Idea for arclength - add up little line segments



length of small segment at t?



bit of arclength near t (x(t), y(t))



$$\sqrt{x'(t)^2\Delta t^2 + y'(t)^2\Delta t^2} = \sqrt{x'(t)^2 + y'(t)^2}\,\Delta t$$

Arclength from t=a to t=b given by

$$\int^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$\int_a^b \underbrace{\sqrt{x'(t)^2 + y'(t)^2}\, dt}_{ds}$$

$x(t) = t \qquad y(t) = t^{3/2} \qquad 0 \le t \le 3$

$$\text{Arclength} = \int_0^3 \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$x'(t) = 1 \qquad y'(t) = \frac{3}{2} t^{1/2}$

$x'(t)^2 = 1 \qquad y'(t)^2 = \frac{9}{4} t$

$$\int_0^3 \sqrt{1 + 9/4\, t}\, dt = \frac{4}{9} \int_1^{31/4} \sqrt{u}\, du$$

$u = 1 + 9/4\, t \qquad t = 0 \rightsquigarrow u = 1$

$du = \frac{9}{4} dt \qquad t = 3 \rightsquigarrow u = 31/4$

$$\frac{4}{9}\left[\frac{2}{3} u^{3/2}\right]_1^{31/4} = \frac{4}{9} \cdot \frac{2}{3} \left(\left(\frac{31}{4}\right)^{3/2} - 1^{3/2} \right)$$

Ex: arclength around circle of radius 1.

$(\overset{x(t)}{\cos t}, \overset{y(t)}{\sin t}) \qquad 0 \le t \le 2\pi$

$$\int^{2\pi} \sqrt{\ldots}\, dt$$

$$\int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

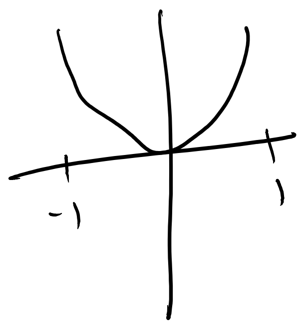
$x'(t) = -\sin t \qquad y'(t) = \cos t$

$x'(t)^2 = \sin^2 t \qquad y'(t)^2 = \cos^2 t$

$$\int_0^{2\pi} \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1}}\, dt = \int_0^{2\pi} dt = t \Big]_0^{2\pi} = 2\pi$$

Ex

$y = x^2$



$-1 \le x \le 1$

$x = t$

$y = x^2 = t^2$

$\boxed{t = x}$!

$$\int \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$x(t) = t = x$

$x'(t) = 1$

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \leftarrow \text{same formula if } \boxed{t = x}$$

$y = x^2 \quad -1 \le x \le 1 \qquad \frac{dy}{dx} = 2x$

$y = x \quad -1 \le x \le 1$ $\qquad \frac{dy}{dx} = 2x$

$$\int_{-1}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-1}^{1} \sqrt{1 + 4x^2}\, dx$$

Stuck for now.