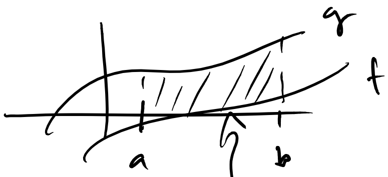
Lecture 4: areas and volumes

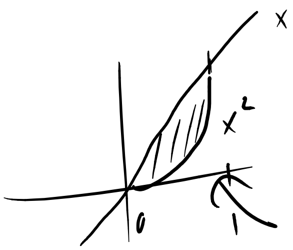
Tuesday, August 26, 2014 10:44 AM

- areas between curves
- volumes of solids via cross-sections



Area is given by $\int_a^b (g(x) - f(x))\,dx$

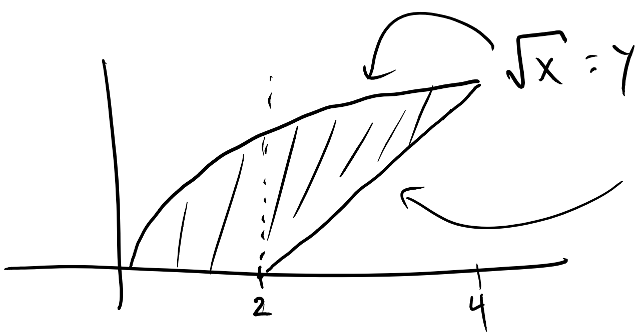
ex



$$\int_0^1 (x - x^2)\,dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

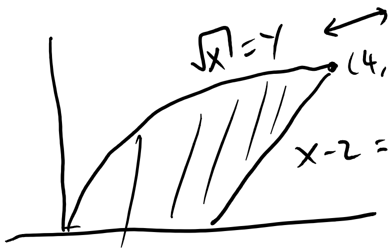
ex



$$\int_0^2 (\sqrt{x} - 0)\,dx + \int_2^4 (\sqrt{x} - (x-2))\,dx$$

$$\left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_2^4$$

or:



$\sqrt{x} = x - 2$

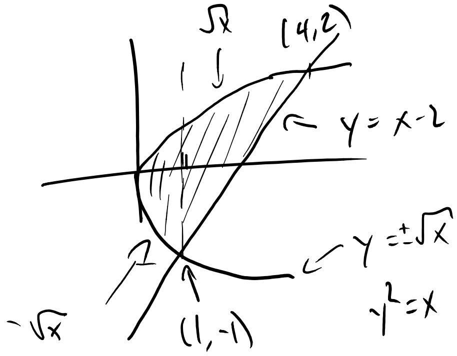
$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

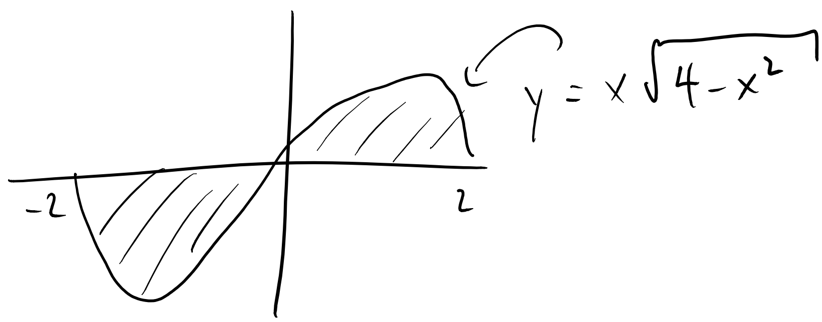
$$\int_{y=0}^{y=2}\left((y+2)-y^2\right)dy = \left[\tfrac{1}{2}y^2+2y-\tfrac{1}{3}y^3\right]_0^2$$

$$= \tfrac{1}{2}2^2+2\cdot 2-\tfrac{1}{3}\cdot 2^3 = \tfrac{10}{3}$$



$$\int_0^1\left(\sqrt{x}-(-\sqrt{x})\right)dx$$

$$+\int_1^4\left(\sqrt{x}-(x-2)\right)dx$$



$$\int_{-2}^{0}\left(0-x\sqrt{4-x^2}\right)dx + \int_0^2\left(x\sqrt{4-x^2}-0\right)dx$$

$$-\int_{-2}^{0}x\sqrt{4-x^2}\,dx$$

$u=4-x^2$

$$= +\tfrac{1}{2}\int_{u=0}^{u=4}u^{1/2}\,du + \left(-\tfrac{1}{2}\right)\int_{u=4}^{u=0}u^{1/2}\,du$$

scratch work

$$\int x\sqrt{4-x^2}\,dx = -\tfrac{1}{2}\int\sqrt{u}\,du$$

$u = 4-x^2$ $\quad x=-2 \to u=0$

$du = -2x\,dx$ $\quad x=0 \to u=4$

$-\tfrac{1}{2}du = x\,dx$ $\quad x=2 \to u=0$

$$= +\frac{1}{2}\int_{u=0} u^{1/2}\,du + \left(-\frac{1}{2}\right)\int_{u=4} u^{1/2}\,du \qquad \left[\; -\frac{1}{2}du = x\,dx \qquad x=2 \to u=0 \; \right]$$

$$= \frac{1}{2}\int_0^4 u^{1/2}\,du + \frac{1}{2}\int_0^4 u^{1/2}\,du$$

$$= \left(\frac{1}{2}+\frac{1}{2}\right)\int_0^4 u^{1/2}\,du = \int_{u=0}^{u=4} u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big]_0^4$$

$$\frac{2}{3}\left(4^{3/2}-0\right)$$

$$\frac{2}{3}\left(\left(\sqrt{4}\right)^3\right) = \frac{2}{3}8$$

$$= \frac{16}{3}$$

Actually: $x\sqrt{4-x^2} = y$ is <u>odd</u>

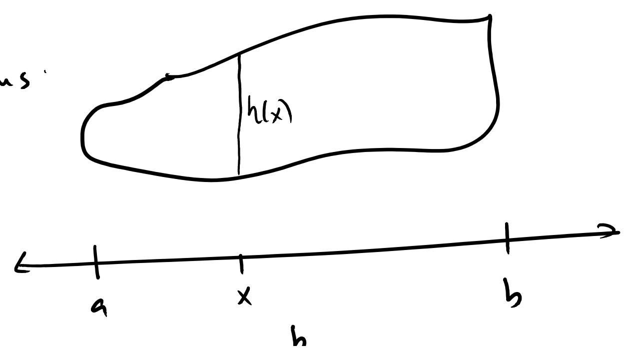
odd: $f(-x) = -f(x)$

$$f(-x) = (-x)\sqrt{4-(-x)^2} = -x\sqrt{4-x^2}$$

$$-f(x) = -x\sqrt{4-x^2}$$ // ! yes.

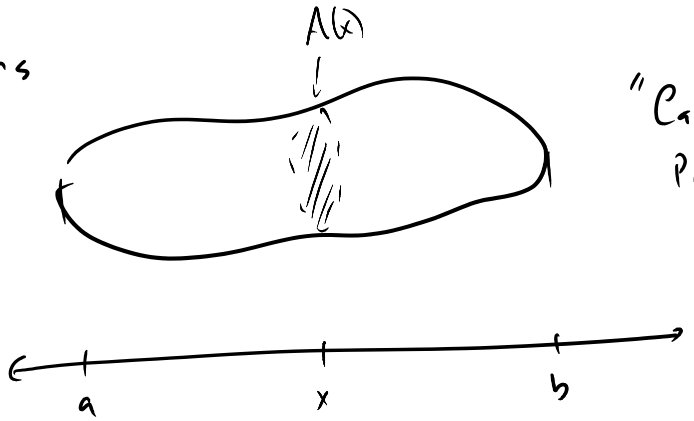
<u>Volumes & cross-sections</u>

2-dims:



$$\text{Area} = \int_a h(x)\,dx$$

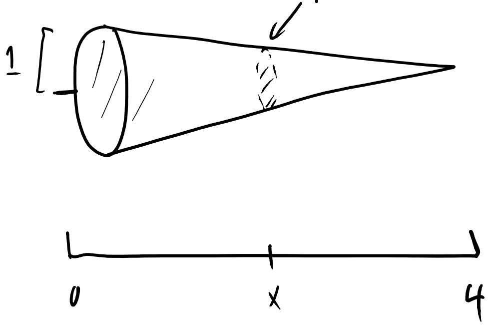
3-dims



"Cavelleri's Principle"

$$\text{Vol} = \int_a^b A(x)\,dx$$

$A(x) = ?$ disc so $A = \pi r^2$



$r = ?$

liner function,

determines $r(x)$ $\begin{cases} \text{starts at } 1 & x=0 \\ \text{ends at } 0 & x=4 \end{cases}$

$r(x) = mx + b$

$r(x) = -\frac{1}{4}x + 1$

$$V = \int_0^4 A(x)\,dx = \int_0^4 \pi\, r(x)^2\,dx$$

$$= \int_0^4 \pi \left(-\frac{1}{4}x + 1\right)^2 dx$$

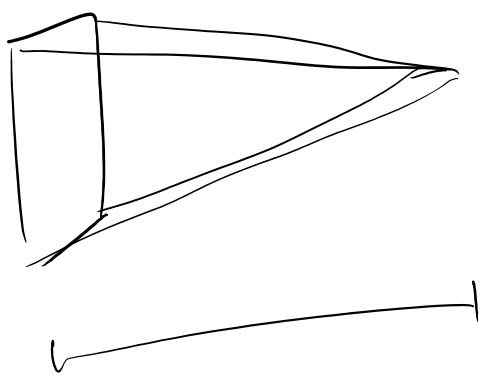
$$= \pi \int_0^4 \left(\frac{1}{16}x^2 - \frac{1}{2}x + 1\right) dx$$

$$= \pi \left[\frac{1}{16}\,\frac{1}{3}x^3 - \frac{1}{4}x^2 + x\right]_0^4$$

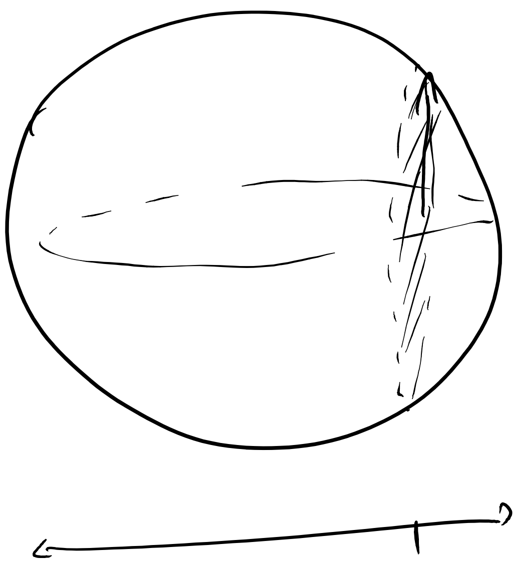
$$= \pi \left[\frac{1}{16} \frac{1}{3} x - \frac{1}{4} x + x \right]_0$$

$$= \pi \left[\frac{1}{16} \frac{1}{3} 4^2 \cdot 4 - \frac{1}{4} 4 \cdot 4 + 4 \right]$$

$$= \pi \left(\frac{4}{3} - 4 + 4 \right) = \pi \frac{4}{3}$$



similar.



$$V = \int_{-1}^{1} A(x)\, dx$$

$$A(x) = \pi\, r(x)^2$$

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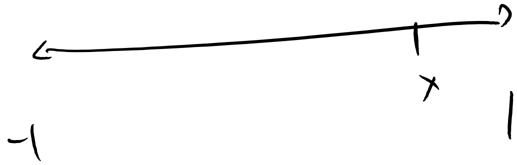
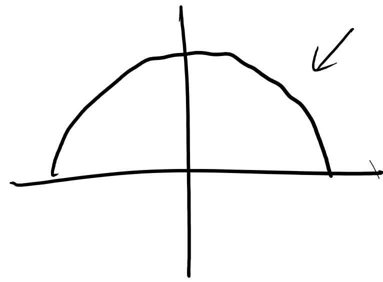


$r(x) = ?$



$x^2 + y^2 = 1$

$y = \sqrt{1-x^2}$

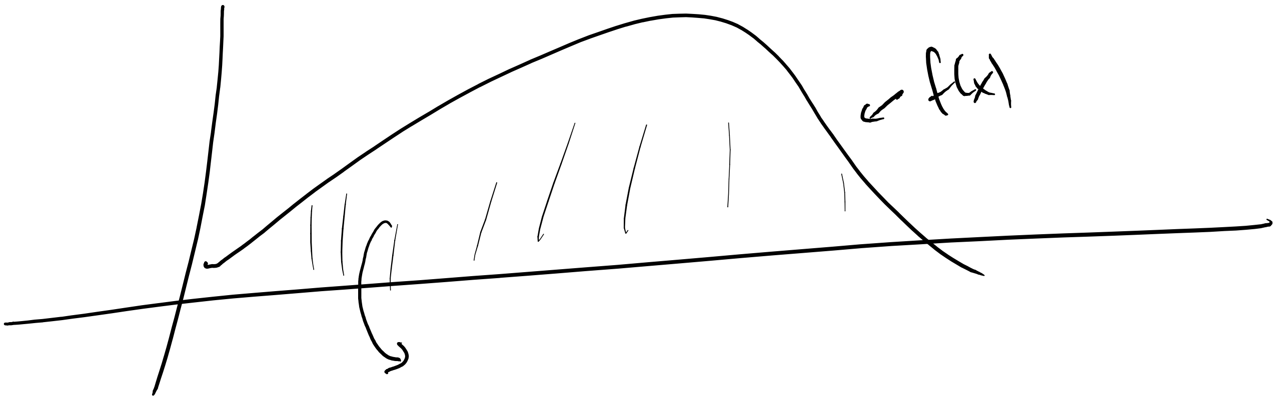
$r(x)$

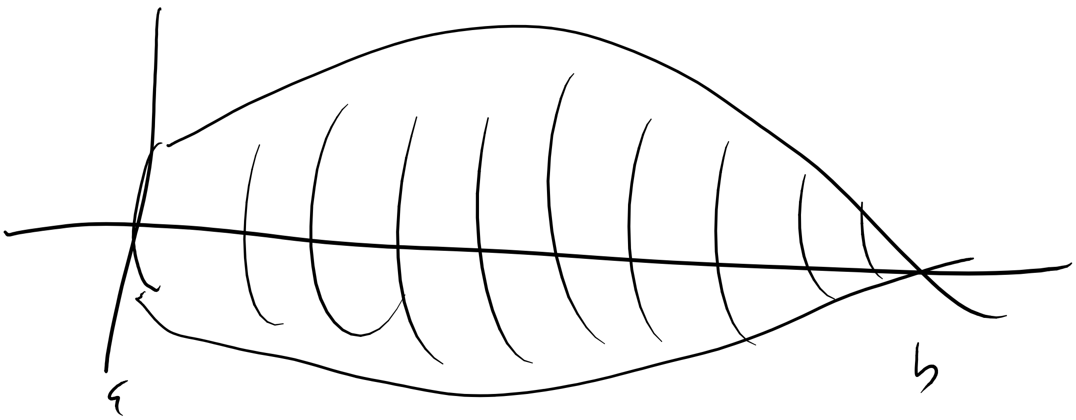
$$A(x) = \pi\left(1-x^2\right)$$

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi\left(1-x^2\right)dx = \pi \int_{-1}^{1} \left(1-x^2\right)dx$$

$$= \pi\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \pi\left[\left(1 - \frac{1}{3}1^3\right) - \left(-1 - \frac{1}{3}(-1)^3\right)\right]$$

$$= \pi\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right] = \frac{4}{3}\pi$$





nice thing = x-sections are discs w/ radius f(x)

$$V = \int_a^b \pi r(x)^2 \, dx = \int_a^b \pi f(x)^2 \, dx$$