Lecture 31: Practice with convergence; Taylor Series

Thursday, November 6, 2014 12:30 PM

Warm-ups

Which of the following series converge absolutely/converge conditionally or diverge. Bonus: what value do they converge to?

1. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$
2. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$
3. $\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{2}\right)^n$
4. $\sum_{n=2}^{\infty} 3 \cdot \left(\frac{1}{3}\right)^n$

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1. Step 1: does it converge absolutely?

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$\downarrow$ integral test

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \int_1^{\infty} x^{-1/2}\,dx \cdots = \infty$$

diverges!

does not absolutely converge.

converge? Alt. series test. 2: (could have done this first)

$$-\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}$$

Converges if $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ this is true!

$$\text{"} \lim_{n\to\infty}\sqrt{\frac{1}{n}}=\sqrt{\lim_{n\to\infty}\frac{1}{n}}=\sqrt{0}=0.$$

$\Rightarrow$ converges conditionally

2: $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$\swarrow$ $\int$ test

converges

$$\int_1^{\infty}x^{-2}\,dx=\lim_{t\to\infty}\left[\frac{1}{-2+1}x^{-2+1}\right]_1^t$$

$$=\lim_{t\to\infty}\left[1-\frac{1}{t}\right]=1$$

Geometric series:

converges absolutely and converges to

$$a+ar+ar^2+ar^3+ar^4+\cdots=\begin{cases}\dfrac{a}{1-r} & |r|<1\end{cases}$$

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \begin{cases} \frac{a}{1-r} & |r|<1 \\ \text{diverges} & |r|\geq 1 \end{cases}$$

$\cdot r \quad \cdot r \quad \cdot r$

$$\sum_{n=2}^{\infty} 3\left(\tfrac{1}{3}\right)^n = 3\cdot\left(\tfrac{1}{3}\right)^2 + 3\cdot\left(\tfrac{1}{3}\right)^3 + 3\cdot\left(\tfrac{1}{3}\right)^4 + \cdots$$

$a \qquad \cdot\frac{1}{3} \qquad \cdot\frac{1}{3}$

$$= \frac{3\cdot\left(\frac{1}{3}\right)^2}{1-\frac{1}{3}} = \frac{1/3}{2/3} = \frac{1}{2}$$

Warm up 2:

Radius of convergence

For which values of x does the series converge?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} x^n$

2. $\displaystyle\sum_{n=1}^{\infty} 2^n \cdot n (x-1)^n$

3. $\displaystyle\sum_{n=1}^{\infty} n!\, x^n$

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1. (abs. converge) $\lim_{n\to\infty} \dfrac{\left|\frac{1}{(n+1)!}x^{n+1}\right|}{\left|\frac{1}{n!}x^n\right|} = \lim_{n\to\infty} \frac{1}{(n+1)!}\cdot\frac{n!}{1}\cdot\frac{|x^{n+1}|}{|x^n|}$

$$= \lim_{n\to\infty} \frac{\cancel{n(n-1)(n-2)(n-3)\cdots 3\cdot2\cdot1}}{(n+1)\cancel{(n)(n-1)(n-2)\cdots 3\cdot2\cdot1}}\frac{|x^{n+1}|}{|x^n|}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}|x| = 0 = \rho$$ converges absolutely everywhere.

2. $\lim_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \dfrac{|2^{n+1}(n+1)(x-1)^{n+1}|}{|2^n n(x-1)^n|}$

$$= \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\right|\left(\frac{n+1}{n}\right)\left|\frac{(x-1)^{n+1}}{(x-1)^n}\right|$$

$$\lim_{n\to\infty} 2\cdot\frac{n+1}{n}|x-1| = 2|x-1| = \text{"}\rho\text{"}$$

Q: For which x values is $\rho < 1$?

$2|x-1| < 1$ $\longrightarrow$ $-1 < 2x-2 < 1$

$|x-1| < \frac{1}{2}$

distance between x & 1 is less than $\frac{1}{2}$

$\frac{1}{2} < x < \frac{3}{2}$. absolutely converges

diverges $x > \frac{3}{2}$ $x < \frac{1}{2}$

$x = \frac{1}{2}, \frac{3}{2}$?

↳ just write it down ⟹ diverges for $x = \frac{1}{2}, \frac{3}{2}$

answer: x in $\left(\frac{1}{2}, \frac{3}{2}\right)$

Given a function $f(x)$ know $f(x)$ near $a = x$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''}{3 \cdot 2}(x-a)^3 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$ "Taylor Series for $f(x)$ at $x = a$"

<u>ex:</u> $f(x) = e^x$, $a = 0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$\sum_{n=0}^{\infty} \frac{e^0}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

in fact, for every x, $\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n}$

-1

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}\cdot(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}\cdot(-1) = 2(1-x)^{-3}$$

$$f'''(x) = -3\cdot 2(1-x)^{-4}(-1) = 3\cdot 2(1-x)^{-4}$$

$$f''''(x) = -4\cdot 3\cdot 2\,(1-x)^{-5}(-1) = 4\cdot 3\cdot 2(1-x)^{-5}$$

$$f^{(n)}(x) = n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1(1-x)^{-(n+1)}$$

$$= n!\,(1-x)^{-(n+1)}$$

Taylor series

$x=0$

$a=$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{n!\,(1-0)^{-(n+1)}}{n!}x^n$$

$$= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

converges if $|x|<1$

to $\frac{1}{1-x}$

$$\sin(2) \approx \sum_{n=0}^{\infty} \frac{(\sin x)^{(n)}\big|_{x=0}}{n!}(2)^n$$

$$\frac{\sin 0}{\cancel{0!}} 2^0 + \frac{\cos 0^1}{1!} (2)^1 - \frac{\sin 0}{\cancel{2!}} (2)^2 - \frac{\cos 0^1}{3!} 2^3$$

$$2 - \frac{1}{6} \cdot 8$$