Lecture 29: the ratio and root tests

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Useful tests for convergence of series:

Ratio & Root tests:

Ratio test: If $\sum_{n=1}^{\infty} a_n$ is a series w/ positive terms (each $a_n > 0$)

and if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ then

- series converges if $\rho < 1$
- series diverges if $\rho > 1$
- "test is inconclusive" if $\rho = 1$

ex. $\sum_{n=1}^{\infty} \frac{n^4}{4^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)^4}{4^{n+1}}\right)}{\left(\frac{n^4}{4^n}\right)} = \lim_{n\to\infty} \frac{(n+1)^4 \, 4^n}{4^{n+1} \, n^4}$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(\frac{n+1}{n}\right)^4$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right)^4 = \frac{1}{4}(1)^4 = \frac{1}{4}$$

since $\frac{1}{4} < 1$, series converges.

Intuition $\rightsquigarrow$ $a_{n+1} \approx \frac{1}{4} a_n \approx \frac{1}{4} \cdot \frac{1}{4} a_{n-1} \approx \frac{1}{4} \cdot \frac{4}{4} \cdot \frac{1}{4} a_{n-2}$

eventually a_n's look close to a geometric series

$a_n \approx a_0 \cdot \rho^n$ $\rho = \frac{1}{4} \rightsquigarrow$ geom. converges. saddles a_n

Practice:

decide which converge:

- $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n \cdot 3^{n-1}}$
- $\sum_{n=1}^{\infty} \frac{n+1}{3^n}$
- $\sum \frac{(n-1)!}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{2^{n+1+1}}{(n+1)3^{n+1-1}}\right)}{\left(\frac{2^{n+1}}{n \cdot 3^{n-1}}\right)} = \lim_{n\to\infty} \frac{2^{n+2}}{(n+1)3^n} \cdot \frac{n \cdot 3^{n-1}}{2^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{2 \cdot \cancel{2^{n+1}}}{\cancel{2^{n+1}}}\right) \cdot \left(\frac{\cancel{3^{n-1}}}{3 \cdot \cancel{3^{n-1}}}\right) \cdot \left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{2}{3} \cdot \left(\frac{n}{n+1}\right)$$

$$= \frac{2}{3} \cdot 1 = \frac{2}{3}$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \qquad \lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1} = 1$$

$\frac{2}{3} < 1 \Rightarrow$ series converges!

$$\sum \frac{n+1}{3^n} \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{n+2}{3^{n+1}}\right)}{\left(\frac{n+1}{3^n}\right)}$$

$$= \lim_{n\to\infty} \frac{3^n}{3^{n+1}} \cdot \frac{n+2}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{3} \cdot \left(\frac{n+2}{n+1}\right) = \frac{1}{3} < 1$$

$\Rightarrow$ series converges!

$$\sum \frac{(n-1)!}{(n+1)^2} \qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1-1)!}{(n+1+1)^2}\right)}{\left(\frac{(n-1)!}{(n+1)^2}\right)}$$

$$= \lim_{n\to\infty} \frac{n!}{(n-1)!} \cdot \frac{(n+1)^2}{(n+2)^2} = \lim_{n\to\infty} \frac{n \cdot \cancel{(n-1)}\cancel{(n-2)}\cancel{(n-3)}\cdots\cancel{(1)}}{\cancel{(n-1)}\cancel{(n-2)}\cancel{(n-3)}\cdots\cancel{(1)}} \left(\frac{n+1}{n+2}\right)^2$$

$$= \lim_{n\to\infty} \underbrace{n}_{\infty} \cdot \underbrace{\left(\frac{n+1}{n+2}\right)^2}_{1} = \infty \qquad \Rightarrow \text{series diverges!}$$

big $\cdot$ 1 is big

Root test (Alternative to ratio test) ("alternative to $\int$ test)

Given $\sum_{n=1}^{\infty} a_n$ and if $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$

then
- converges if $\rho < 1$
- diverges if $\rho > 1$
- inconclusive if $\rho = 1$

Idea: if $\lim \sqrt[n]{a_n} = \rho$

means (n big) $\sqrt[n]{a_n} \approx \rho$

$a_n \approx \rho^n$ geometric

examples

$$\sum_{n=1}^{\infty}\left(\frac{2n-3}{4n+1}\right)^n$$

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n-3}{4n+1}\right)^n}=\lim_{n\to\infty}\frac{2n-3}{4n+1}$$

$$=\frac{2}{4}=\frac{1}{2}$$

Reminder

$$\lim_{n\to\infty}\frac{2n-3}{4n+1}\cdot\frac{1/n}{1/n}=\lim_{n\to\infty}\frac{2-3\cdot\frac{1}{n}}{4+1/n}=\frac{2}{4}=\frac{1}{2}$$

or $$\lim_{x\to\infty}\underbrace{\frac{2x-3}{4x+1}}_{\text{"}\frac{\infty}{\infty}\text{"}}\underset{\text{l'hop}}{=}\lim_{x\to\infty}\frac{2}{4}=\frac{2}{4}=\frac{1}{2}$$

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$$

$$\lim_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}}$$

$$\parallel$$

$$\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)^{n^2}\right)^{1/n}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n^2/n}$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=L$$

"1^∞" $\rightsquigarrow$ $\ln L=\lim_{n\to\infty}\ln\left(1-\frac{1}{n}\right)^n$

$$\ln L = \lim_{n\to\infty} n\ln\left(1-\frac{1}{n}\right)$$

$$= \quad \infty\cdot 0$$

$$\lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{1/x} \underset{\text{l'hop}}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{\left(1-\frac{1}{x}\right)}\right)\cdot\frac{1}{x^2}}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{-1}{1-\frac{1}{x}} = -1$$

$$\ln L = -1 \qquad L = e^{-1} = \frac{1}{e} < 1$$

series converges

$$\sum_{n=1}^{\infty} n x^n$$ for which x does this converge?

$$x + 2x^2 + 3x^3 + \cdots$$

Ratio

$$\lim_{n\to\infty} \frac{(n+1)x^{n+1}}{(n)x^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)x = x$$

$x<1$ conv $\quad x>1$ dvge $\quad x=1$?