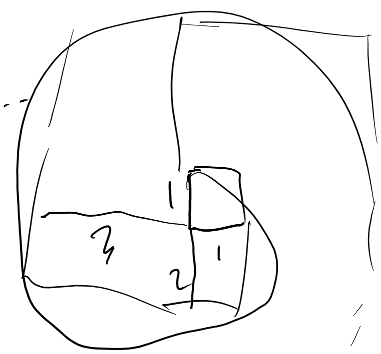
Lecture 28: Radius of convergence and Fibonacci interlude

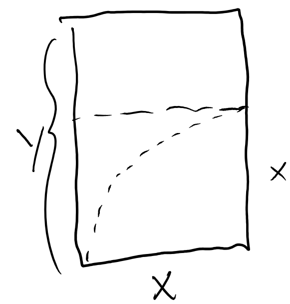
Wednesday, October 29, 2014 1:13 PM

Fibonacci:

1, 1, 2, 3, 5, 8, 13, 21

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$





$$\frac{x}{y} = \frac{y-x}{x} \qquad x = 1$$

$$\frac{1+\sqrt{5}}{2} = \mu \quad \text{"golden ratio"}$$

$$\frac{1}{y} = y - 1$$

$$y^2 - y - 1 = 0$$

$$y = \frac{1 \pm \sqrt{1+4}}{2}$$

$$y = \frac{1 \pm \sqrt{5}}{2}$$

Main Important facts about power series

Recall: a power series is an expression

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots$$ "power series about $x=0$"

ex: $1 + x + x^2 + x^3 + \cdots$ geom. series

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n$$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

"power series about $x=c$"

ex: $1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots$

$$\sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \qquad f(0) = 1$$

$$f\left(\tfrac{1}{2}\right) = 1 - \left(-\tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^3 \cdots$$

$$= 1 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{4}\right) + \tfrac{1}{8} + \cdots$$

$$= 2$$

<u>Definition / Theorem</u> "Radius of convergence"

Given a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ then

Given a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ then one of the following things is true

Either:

1) There is some real number $R>0$ "radius of convergence" such that the series converges when $|x-c|<R$ & diverges when $|x-c|>R$

or 2) Converges for every x ("$R=\infty$")

or 3) Only converges at $x=c$, diverges everywhere else ("$R=0$")

example: $\sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots = \frac{1}{1-x}$

radius of convergence of this is 1.

if $|x|<1$ converges, $|x|>1$ diverges.

$$1+3x+(3x)^2+(3x)^3+\cdots = \sum_{n=0}^{\infty}(3x)^n$$

geom. w/ $r=3x$

$|r|<1 \qquad |3x|<1$

$|x|<\frac{1}{3}$ radius of conv.

$f(x)$

$|r|<1 \qquad |3x|<1$

$|x|<\frac{1}{3}$ radius of conv.

Important fact:

Suppose $\overset{f(x)}{\overset{\shortparallel}{\sum_{n=0}^{\infty} a_n (x-c)^n}}$ $\quad \overset{g(x)}{\overset{\shortparallel}{\sum_{n=0}^{\infty} b_n (x-c)^n}}$

both converge for $|x|<R$ (less than both radii of convergence)

then

- $$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + C$$

$$= \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}(x-c)^n + C$$

- $$\frac{d}{dx} f(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$= \sum_{n=0} a_{n+1}(n+1)\, x^n$$

$$1+x+x^2+\cdots = \frac{1}{1-x} \qquad |x|<1$$

$$1+(-x)+(-x)^2+(-x)^3+\cdots = \frac{1}{1-(-x)} \qquad |x|<1$$

$$1-x+x^2-x^3+x^4+\cdots = \frac{1}{1+x} \qquad |x|<1$$

$$x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5\cdots = \ln|1+x| + C$$

$$= \ln(1+x) + C \qquad |x|<1$$

$0 \quad = \ln \underset{0}{1} + C \qquad x=0$

$C=0$

$$\boxed{\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \cdots \quad |x|<1}$$

<u>also</u>: $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} b_n(x-c)^n$

$$(f+g)(x) = \sum_{n=0}^{\infty} (a_n+b_n)(x-c)^n$$

$(fg)(x) =$ "foil"

<u>ex</u>: $\frac{1}{1-x} \cdot \frac{1}{1-x}$

$= (1+x+x^2+x^3+\cdots)(1+x+x^2+x^3+\cdots)$

$\sim 1+2x+3x^2+4x^3+5x^4 \cdots$

const: $1 \cdot 1 = 1$

x-term: $1 \cdot x + x \cdot 1 = 2x$

x^2-term: $1 \cdot x^2 + x \cdot x + x^2 \cdot 1 = 3x^2$

<u>Fibonacci Series!</u>

intrested in series $a_0=1 \quad a_1=1 \quad a_2=2, \; a_3=3$

$a_4 = 5$

$a_{n+1} = a_n + a_{n-1}$

Consider $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$= 1 + 1 \cdot x + 2 \cdot x^2 + 3 \cdot x^3 + 5x^4 + 8x^5 + \cdots$

$f(x) = 1 + 1x + 2x^2 + 3x^3 + 5x^4 + 8x^5$

$xf(x) = \quad 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 \cdots$

$+$

$x^2 f(x) = \quad 1x^2 + 1x^3 + 2x^4 + 3x^5 + 5 \cdots$

$x^2 f(x) + x f(x) = f(x)$ solve for $f(x)$

$$f(x) = \frac{1}{x^2 + x - 1}$$

$$x^2 + x - 1 = \left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)$$

$\frac{-1 \pm \sqrt{5}}{2}$ $\quad \alpha = \frac{-1+\sqrt{5}}{2}$ $\quad \beta = \frac{-1-\sqrt{5}}{2}$

$$\frac{1}{x^2+x-1} = \frac{A}{x+\alpha} + \frac{B}{x+\beta} = \frac{1}{\sqrt{5}}\left(\frac{-1}{x+\alpha} + \frac{1}{x+\beta}\right)$$

$$= Ax + \beta A + Bx + \alpha B = 1$$

$$(A+B)x + \beta A + \alpha B = 1$$

$$A+B=0 \qquad B=-A$$

$$\beta A - \alpha A = 1$$

$$(\beta - \alpha) = \frac{1}{A} \qquad A = \frac{1}{\beta-\alpha} = -\frac{1}{\sqrt{5}}$$

$$B = \frac{1}{\sqrt{5}}$$