# Lecture 27: Power Series

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Given some function $f(x)$ want to approximate $f(x)$ by a polynomial

$f(x) \approx a_0$ const (deg 0 poly)

$f(x) \approx a_0 + a_1 x$ linear (deg 1 poly)

$f(x) \approx a_0 + a_1 x + a_2 x^2$ quad

$\vdots$

$f(x) \approx a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Notion of a power series:

A power series is an expression of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

example:

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

does this give an expression of some function?

plug in a value for x, do I get an answer?

Problems:

Given $f(x) = 1 + x + x^2 + \cdots$

Find (if they exist)

$f(0)$, $f(1)$, $f(-1)$, $f(\frac{1}{2})$, $f(2)$, $f(-\frac{1}{2})$

$f(0) = 1$, $f(1) =$ d.n.e. $(\infty)$, $f(-1) =$ d.n.e., $f(\frac{1}{2}) = 2$, $f(2) =$ d.n.e. $(\infty)$, $f(-\frac{1}{2}) = \frac{2}{3}$

$$f(-\tfrac{1}{2}) = 1 + (-\tfrac{1}{2}) + (-\tfrac{1}{2})^2 + (-\tfrac{1}{2})^3 + \cdots$$

$$f(x) = 1 + x + x^2 + \cdots = \begin{cases} \frac{1}{1-x} & \text{if } |x| < 1 \\ \text{undefined} & \text{else} \end{cases}$$

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$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots$ $\qquad a_n = \left(\frac{1}{2}\right)^n$

$f(x) = \left(\frac{1}{2}\right)^x$

$\int$ test: $\sum_{n=0}^{\infty} a_n$ converges if and only if $\int^{\infty} f(x)\,dx$

$$\sum_{n=0}$$

$$\int_0^\infty f(x)\,dx \text{ converges}$$

$$\int_0^\infty \left(\tfrac{1}{2}\right)^x dx = \lim_{t\to\infty} \int_0^t \left(\tfrac{1}{2}\right)^x dx = \lim_{t\to\infty} \int_0^t e^{x\ln\frac{1}{2}}\,dx$$

$$u = x\ln\tfrac{1}{2}$$

$$du = \ln\tfrac{1}{2}\,dx$$

$$a^x = e^{x\ln a}$$

$$= \lim_{t\to\infty} \frac{1}{\ln\frac{1}{2}} \int_0^t e^u\,du$$

$$= \lim_{t\to\infty} \frac{1}{\ln\frac{1}{2}} \left[e^{x\ln\frac{1}{2}}\right]_0^t$$

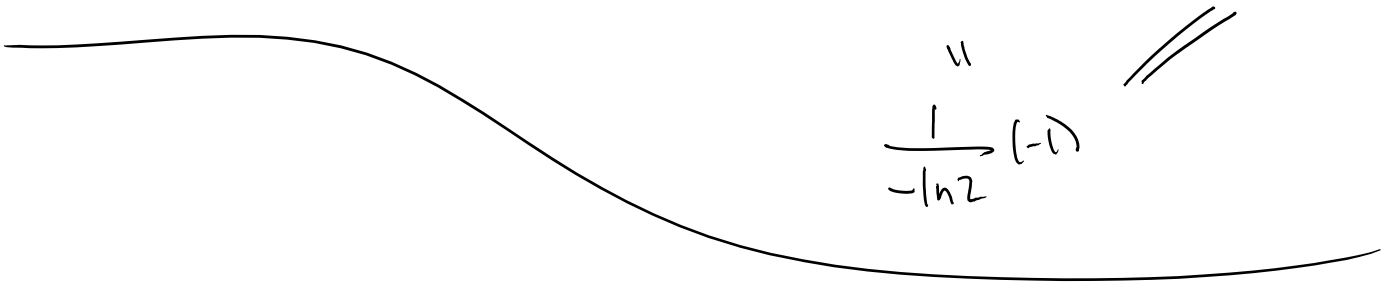
$$= \lim_{t\to\infty} \frac{1}{\ln\frac{1}{2}} \left(e^{t\ln\frac{1}{2}} - 1\right)$$

$\ln\frac{1}{2} < 0$ since $\frac{1}{2} < 1$

$t\ln\frac{1}{2} \rightsquigarrow$ big & negative

$e^{t\ln\frac{1}{2}} \rightsquigarrow 0$

$$= \frac{1}{\ln\frac{1}{2}}(-1) = \frac{1}{\ln 2}$$



$$\frac{1}{-\ln 2}(-1)$$

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \cdots$$

which of these make sense?

$$f(0) = 1$$

$$f\left(\tfrac{1}{3}\right) = 1 + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3^2} + 4 \cdot \frac{1}{3^3} + \cdots$$

$$f(1) = \text{d.n.e.}$$

$$a_n = (n+1) \cdot \frac{1}{3^n}$$

$$f(x) = (x+1)\,\frac{1}{3^x}$$

$$\int (x+1)\,\frac{1}{3^x}\,dx$$

$$\int (x+1)\,e^{-x\ln 3}\,dx$$

$$a_0 = (0+1)\,\frac{1}{3^0} = 1$$

$$a_1 = (1+1)\,\frac{1}{3^1} = 2 \cdot \frac{1}{3}$$