Lecture 26: Series and convergence

Tuesday, October 28, 2014 12:29 PM

Geometric Series:

$$a + ar + ar^2 + ar^3 + \cdots + ar^n = \frac{a(1-r^{n+1})}{1-r}$$

$$\sum_{i=0}^{n} ar^i = \frac{a(1-r^{n+1})}{1-r}$$

taking the limit: $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$ (sometimes) (if $|r|<1$)

$$\sum_{i=0}^{\infty} ar^i = \lim_{n\to\infty} \sum_{i=0}^{n} ar^i$$

actually $\lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r}$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

$\left(\frac{1}{2}\right)^{n+1} \to 0$

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i = \lim_{n\to\infty} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{(1/2)} = 2$$

<u>Warm up:</u> Find the following limits $\lim_{n\to\infty} a_n$

1. $a_n = \dfrac{1-2n}{1+2n}$ $n \to \infty$

2. $a_n = \sqrt[n]{n^2}$ $n \to \infty$

3. $a_n = 5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + \dots + \frac{5}{3^n}$ $n \to \infty$

4. $a_n = \dfrac{n!}{n^n}$

---

1. $\lim_{n\to\infty} \dfrac{1-2n}{1+2n} = -1$

$$\lim_{x\to\infty} \frac{1-2x}{1+2x} \overset{\text{l'hop}}{=} \lim_{x\to\infty} \frac{-2}{2} = -1$$

2. $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} (n^2)^{1/n} = \lim_{n\to\infty} n^{2/n} = 1$ "∞^0" situation

$$\lim X^{2/x} = L$$

$$\lim_{x\to\infty} X = L$$

$$\ln\left(\lim_{x\to\infty} x^{2/x}\right) = \ln L$$

$$\parallel$$

$$\lim_{x\to\infty} \ln(x^{2/x}) = \lim_{x\to\infty} \frac{2}{x}\ln x$$

$$= \lim_{x\to\infty} \frac{2\ln x}{x}$$

$$\text{L'hop} \to \lim_{x\to\infty} \frac{2\,{}^{1}\!/_{x}}{1} = 0$$

$$\ln L = 0$$

$$L = e^{0} = 1$$

3. $\left(15/2\right) \Rightarrow \dfrac{a}{1-r} = \dfrac{5}{1-\frac{1}{3}} = \dfrac{5}{\left(\frac{2}{3}\right)} = \dfrac{15}{2}$

$|r| = |\tfrac{1}{3}| < 1$

$$5 + 5/3 + 5/9 + \text{—} + 5/81 = \frac{a(1-r^{n+1})}{1-r}$$

$$5 + 5\cdot 3 + 5\cdot 9 + 5\cdot 27 = \frac{5(1-3^{4})}{1-3}$$

4. $\frac{n!}{n^n} = \frac{n\cdot(n-1)(n-2)\cdots 1}{n\cdot n\cdot n\cdots n} = \underbrace{\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)}_{\text{each} \leq 1}\cdots\left(\frac{1}{n}\right)$

last one is $\frac{1}{n}$

$\Rightarrow 0 \leq \frac{n!}{n^n} \leq 1\cdot 1\cdots 1\cdot\frac{1}{n} = \frac{1}{n}$

Convergence & Divergence of series $\sum_{n=1}^{\infty} a_n$

- Test: if it's a geometric series, converges if $|r|<1$
- terms need to get small!

If $\lim_{n\to\infty} a_n \neq 0$ then series diverges

"n^{th} term test"

ex: $\sum_{n=1}^{\infty} \frac{n}{n+1}$ $\lim_{n\to\infty} \frac{n}{n+1} = 1$ so diverges

doesn't go both ways!

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\frac{1}{n} \to 0$, but series diverges!

• Method 3: Luck

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) \cdots$$

$$\frac{1}{1} \cancel{- \frac{1}{2} + \frac{1}{2}} \cancel{- \frac{1}{3} + \frac{1}{3}} \cancel{- \frac{1}{4} +} \cdots \cancel{\cdots \frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$ $n \to \infty$ this goes to 1.

"Break up fractions & try to cancel"

Useful tools:

If $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$ convergent

then

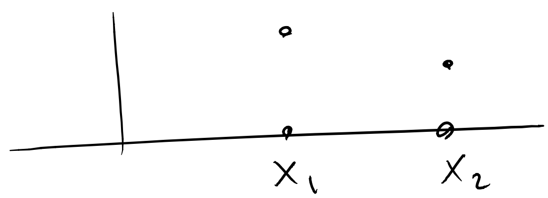
Sum $\sum_{n=1}^{\infty} a_n + b_n = A + B$

Difference $\sum_{n=1}^{\infty} a_n - b_n = A - B$

Const. mult. $\sum_{n=1}^{\infty} c a_n = cA$ c any real #.

Integral test

"Recall" that a function $f(x)$ is called nonincreasing if $f(x_2) \le f(x_1)$ for $x_2 \geq x_1$



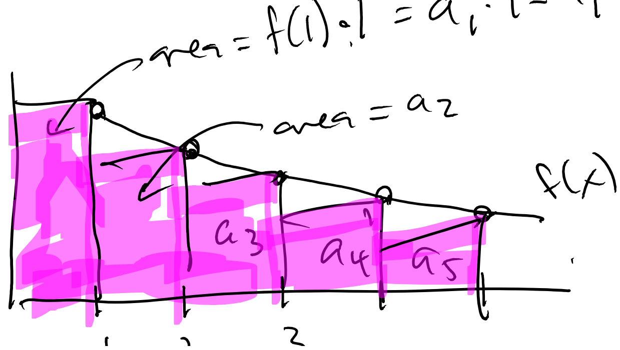
Theorem "The integral test"

Suppose $f(x)$ is nonincreasing & $a_n = f(n)$.

Then either $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)\,dx$ both converge

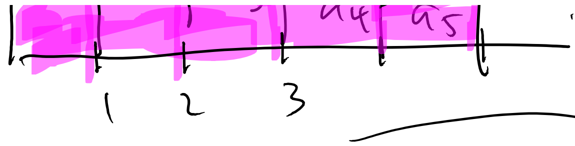
or they both diverge.

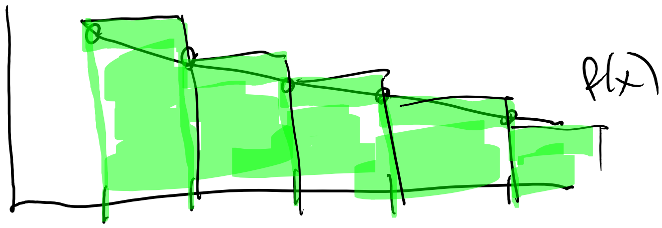
Visual explanation:



$$\sum_{n=1}^{\infty} a_n$$

$$a_1 + \sum_{n=2}^{\infty} a_n$$





$$a_1 + \underbrace{\sum_{n=2}^{\infty} a_n}_{\text{all underneath graph}} \le a_1 + \int_1^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{\infty} a_n =$$

graph is below this!

$$\sum_{n=1}^{\infty} a_n \ge \int_1^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{\infty} a_n \ge \int_1^{\infty} f(x)\,dx \ge \sum_{n=2}^{\infty} a_n$$

$$a_1 + \int_1^{\infty} f(x)\,dx \ge \sum_{n=1}^{\infty} a_n \ge \int_1^{\infty} f(x)\,dx$$

---

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\qquad f(x) = \frac{1}{x^2}$

$$\int^{\infty} \frac{1}{x^2}\,dx = \lim \int^{t} 1/x^2\,dx = \lim \left[-\frac{1}{x}\right]_1^t$$

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t 1/x^2\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{t}-\left(-\frac{1}{1}\right)\right)$$

$$= \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$$

$\int$ converges! so series converges!

$$2 = 1+\frac{1}{1^2} \geq \sum_{n=1}^\infty \frac{1}{n^2} \geq 1$$

$$\left(= \frac{\pi^2}{6}\right)$$

$\sum_{n=1}^\infty \frac{1}{n^p}$ $\quad \int$ test $\rightsquigarrow$ converges if $p>1$

diverges otherwise!

$\sum_{n=1}^\infty \frac{1}{n}$ $\quad \int \frac{1}{x}dx = \ln|x|$

$$\lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}[\ln t - \underset{0}{\underset{\shortparallel}{\ln 1}}]$$

$$= \infty$$