Lecture 25: convergence of sequences and some series

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Two tools: Sandwich/Squeeze theorem, Monotonic sequence theorem

Sandwich theorem:

Given 3 sequences $\{a_n\}, \{b_n\}, \{c_n\}$ with

$a_n \leq b_n \leq c_n$, and if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

then $\lim_{n\to\infty} b_n = L$ also.

Practice: Given "b_n", you need to come up with a_n, c_n's

ex: $\lim_{n\to\infty} \frac{|\cos n|}{n} = 0$ why?

can stick it between 2 things I understand:

$$0 \leq \frac{|\cos n|}{n} \leq \frac{1}{n}$$

Because $\lim_{n\to\infty} 0 = 0$, and $\lim_{n\to\infty} \frac{1}{n} = 0$ the squeeze theorem tells us that $\lim_{n\to\infty} \frac{|\cos n|}{n} = 0$.

$\lim_{n\to\infty} r^n$ r some number

$r=1$

$a_n = 1^n$ $a_1 = 1$

$a_2 = 1$ ... so $\lim_{n\to\infty} 1^n = 1$

$a_3 = 1$

$\lim_{n\to\infty} (-1)^n =$ divergent

$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

$\lim_{n\to\infty} (0.998)^n = 0$

$\lim_{n\to\infty} (1.1)^n = \infty$

$\lim_{n\to\infty} \left(-\frac{1}{3}\right)^n = 0$

$$-\left(\frac{1}{3}\right)^n \le \left(-\frac{1}{3}\right)^n \le \left(\frac{1}{3}\right)^n$$

$$\left(-\frac{1}{3}\right)^n = (-1)^n\left(\frac{1}{3}\right)^n$$

$\lim -\left(\frac{1}{3}\right)^n = -\lim \left(\frac{1}{3}\right)^n = 0$

$\lim \left(\frac{1}{3}\right)^n = 0$

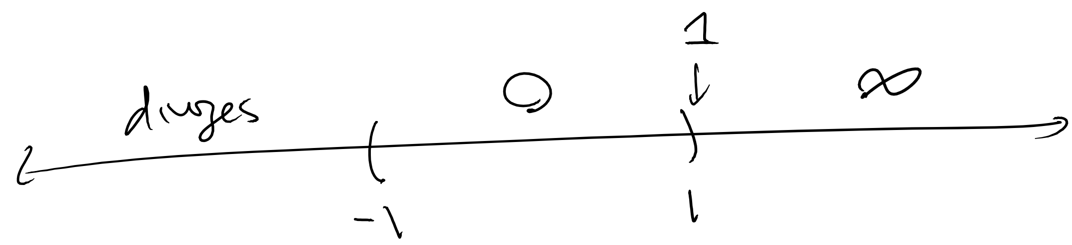
+ squeeze

$\Downarrow$

$\lim_{n\to\infty} \left(-\frac{1}{3}\right)^n = 0$

General fact:

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if } r=1 \\ \text{dne.} & \text{if } r \le -1 \\ \infty & \text{if } r>1 \\ 0 & \text{if } |r|<1 \end{cases}$$



Monotonic Sequence theorem

If $\{a_n\}$ is a sequence which is

monotonic nondecreasing: $a_{n+1} \geq a_n$ all n's.

& bounded: $a_n \leq M$ some fixed M all n's.

then $\lim_{n\to\infty} a_n = L$ some $L \leq M$.

$a_1 = 1$

$a_2 = 1 + \frac{1}{2}|\sin 2|$

$a_3 = 1 + \frac{1}{2}|\sin 2| + \frac{1}{4}|\sin 3|$

$a_4 = 1 + \frac{1}{2}|\sin 2| + \frac{1}{4}|\sin 3| + \frac{1}{8}|\sin 4|$

$a_5 =$ " — — — — $+ \frac{1}{16}|\sin 5|$

$a_5 \geq a_4$ $\quad a_{n+1} \geq a_n$

$a_n \leq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$

" $1 + \frac{1}{2}|\sin 2| + \frac{1}{4}|\sin 3|$

$|\sin 2| \leq 1 \quad |\sin 3| \leq 1 \ \ldots$

$\frac{1}{2}|\sin 2| \leq \frac{1}{2}$

so they converge.

to what? I have no idea.

Series

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ $\quad r = \frac{1}{2}$

$\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n$

$(1 + r + r^2 + \cdots + r^n)(1 - r)$

$= (1 + r + r^2 + \cdots + r^n) - (r + r^2 + r^3 + \cdots + r^{n+1})$

$= 1 - r^{n+1}$

$(1 + r + \cdots + r^n)(1 - r) = 1 - r^{n+1}$

$(1+r+\cdots+r^n)(1-r) =$

$$1+r+\cdots+r^n = \frac{1-r^{n+1}}{1-r}$$

if $r=\frac{1}{2}$, if n gets big RHS gets close to

$$\lim_{n\to\infty} \frac{1-(\frac{1}{2})^{n+1}}{1-(\frac{1}{2})} = \frac{1}{1-\frac{1}{2}} = 2$$

Definition Given a sequence of real #s $\{a_n\}_{n\geq 1}$ an expression of the form $a_1+a_2+a_3+\cdots$ is called a series

a_n is called the n^{th} term of the series

$$\left.\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1+a_2 \\ s_3 &= a_1+a_2+a_3 \\ s_n &= \sum_{i=1}^{n} a_i \end{aligned}\right\}$$

$\{s_n\}$ are called the sequence of partial sums

s_n = nth partial sum

if $\lim_{n\to\infty} s_n = L$ we say the series converges to L (otherwise it diverges).

example $a_n = \left(\frac{1}{3}\right)^n$

series $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{3^n}$

5th partial sum

$S_5 = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5}$

$$\boxed{1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad \text{"geometric series"}}$$

$$S_5 = \frac{1 - \left(\frac{1}{3}\right)^6}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{729}}{1 - \frac{1}{3}} = \frac{\left(\frac{728}{729}\right)}{\left(\frac{2}{3}\right)} = \frac{728}{2} \cdot \frac{3}{729}$$

$$= 364 \cdot \frac{1}{243}$$

$$= \frac{364}{243}$$

Harmonic Series

$a_n = \frac{1}{n}$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n}$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ count the 1/2's.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

count in $1/2$'s.

≥ 1

$\underbrace{}_{1\,1/2}$ $\quad \frac{1}{3}, \frac{1}{4} \geq \frac{1}{4}$

2 more than $\frac{1}{4} \geq \frac{1}{2}$

$\underbrace{}_{2}$

$\underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$

4 thgs each $\geq \frac{1}{8}$

$\geq \frac{1}{2}$

$\underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}$

8 thgs $\geq \frac{1}{16}$

$\geq \frac{1}{2}$