Lecture 23: Sequences!

Tuesday, October 21, 2014 12:26 PM

A sequence (infinite) is an ordered list of real numbers.

ex: $1, 2, 3, 4, \ldots$

$0, 0, 0, 0 \ldots$

$1, 1/2, 1/3, 1/4, 1/5, \ldots$

$1, 2, 4, 8, 16, 32, \ldots$

$3, 1, 7, 2, 9, 4, -3, 7, 2, \ldots$

Notation

$a_1, a_2, a_3 \ldots$ subscript = "index"

$a_1 = 1^{st}$ # analogous to $a_n \leftrightarrow a(n)$

$a_2 = 2^{nd}$ #

$1, 2, 3, 4, 5, \ldots$ $a_1 = 1,\ a_2 = 2,\ a_3 = 3$

$a_n = n$

ex: $1, 2, 4, 8, 16 \ldots$ $a_n = 2^n$

ex: $1, 1/2, 1/3, \ldots$ $a_n = 1/n$

Usually, we'll start w/ index 1

a_1, a_2

sometimes we'll let ourselves start w/ different index

$a_n = 2n \qquad n \geq 5$

$10, 12, 14, \ldots$

Say a_n converges to L if a_n gets as close as we want to L as long as n is sufficiently large.

$\lim_{n \to \infty} a_n = L$ alternate notation $a_n \longrightarrow L$

ex: $\lim_{n \to \infty} 1/n = 0$

if $\lim_{n \to \infty} a_n$ not equal to any L's, we say a_n diverges.

ex: $a_n = (-1)^n \qquad n \geq 0$

$a_0, a_1, a_2, \ldots$

$1, -1, 1, -1, \ldots$

Notation:

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a_n a number

$\{a_n\}$ a sequence

$\{a_n\}_{n=1,2,\ldots} = \{a_n\}_{n\geq 1}$

$\{b_n\}_{n\geq 5}$

$\{c_n\}_{n\geq -2}$

$a_n = n^2$

$a_n = n^2 \quad a_5 = 25$

$\{n^2\}_{n\geq 1} = \{a_n\}_{n\geq 1} = 1, 4, 9, 16, 25, \ldots$

Algebra w/ sequences

$\{a_n\}, \{b_n\}$ sequences, we define

$\{a_n\} + \{b_n\} = \{a_n + b_n\}$

$\{a_n\} - \{b_n\} = \{a_n - b_n\}$

$\{a_n\}\{b_n\} = \{a_n b_n\}$

$k\{a_n\} = \{k a_n\}$ k real #.

$\{a_n\}/\{b_n\} = \{a_n/b_n\}$ if b_n's aren't 0.

if $f(x)$ is any function, $f\{a_n\} = \{f(a_n)\}$.

ex: $\{n^2\} = \{n\}^2$

$\sqrt{\{a_n\}} = \{\sqrt{a_n}\}$

Theorem $\{a_n\}, \{b_n\}$ sequences and $\lim_{n\to\infty} a_n = A$

$\lim_{n\to\infty} b_n = B$

1. $a_n + b_n \longrightarrow A+B$
2. $a_n - b_n \longrightarrow A-B$
3. $a_n b_n \longrightarrow AB$
4. $k a_n \longrightarrow kA$
5. $a_n / b_n \longrightarrow A/B$ (if makes sense)
6. if $f(x)$ is cont. then $f(a_n) \longrightarrow f(A)$

Sturdy facts:

$\frac{1}{n} \to 0$

$a_n \to \infty$ (means a_n gets large as n gets big)

means a_n gets as large as we want if n is sufficiently large.

$n \to \infty$

$e^n \to \infty$

$\sqrt{n} \to \infty$

$\log_{10} n \to \infty$

$\ln n \to \infty$

$\lim_{n\to\infty} n = \infty$

$\lim_{n\to\infty} e^n = \infty$

$$\boxed{\begin{array}{c} a_n \to \infty \\ \Updownarrow \quad a_n > 0 \\ \frac{1}{a_n} \to 0 \end{array}}$$

Relate to functions

Relate to functions

If $f(x)$ is a function, an $a_n = f(n)$

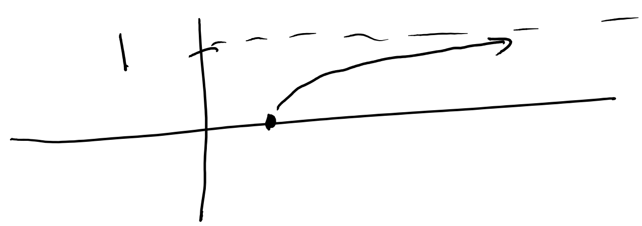
$(\sqrt{x}) \longleftrightarrow (a_n = \sqrt{n})$

then $\lim_{x\to\infty} f(x) = L$ implies $\lim_{n\to\infty} a_n = L$.

Here's what you need to start to remember:

- Horizontal Asymptotes
- L'Hopital's rule

ex: $f(x) = 1 - \frac{1}{x}$ $\qquad \lim_{x\to\infty} f(x) = 1$



$a_n = f(n) = 1 - \frac{1}{n}$ $\qquad a_n \to 1$

ex: $f(x) = x\sin(1/x)$

$\sin(1/x)$

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$$\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{(1/x)}$$

$$= \lim_{t\to 0} \frac{\sin t}{t} = 1$$

$$a_n = n\sin(1/n) \longrightarrow 1$$

L'hopital's rule:

If $f(x), g(x)$ are differentiable &

$\lim_{x\to a} f(x) = \infty$ (or 0)

$\lim_{x\to a} g(x) = \infty$ (or 0)

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

"$\frac{\infty}{\infty}$, $\frac{0}{0}$" $\Rightarrow$ take der. of top & bottom.

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{L'hop.}}{=} \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = 1.$$

$\sin x \to 0$

$x \to 0$

ex: 1. $a_n = \frac{\ln(n)}{n}$ $\qquad \lim_{n\to\infty} a_n = ?$

2. $a_n = \frac{n-1}{n}$ $\qquad$ 3. $a_n = \frac{(\ln n)^2}{n}$

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1. $\ln(n) \to \infty$ $\qquad \frac{\ln x}{x}$ $\qquad \ln x \to \infty$

$n \to \infty$ $\qquad\qquad x \to \infty$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\Rightarrow a_n \to 0$$

2. $$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \frac{n}{n} - \frac{1}{n} = \lim_{n\to\infty} 1 - \frac{1}{n}$$

$$= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1.$$

3. $$\lim_{n\to\infty} \frac{(\ln n)^2}{n}$$

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{\text{l'hop}}{=} \lim_{x\to\infty} \frac{2(\ln x)\,1/x}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}$$

$$\text{“}\frac{\infty}{\infty}\text{”} \qquad = 2 \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

Calculated.

did it)

$$a_n = \frac{2^n}{n!}.$$