Lecture 19: improper integrals

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Q: $\int_{-1}^{1} 1/x \, dx = 0$? (actually divergent)

$$\int_0^1 1/x \, dx = \underset{0}{\underset{\shortparallel}{\ln 1}} - \underset{\text{"}-\infty\text{"}}{\underset{\shortparallel}{\ln 0}} = \infty$$

Improper Integral $\rightsquigarrow$ 0 not in domain so doesn't actually make sense.

to "evaluate" (translate) we write $\int_0^1 1/x \, dx$

$$= \lim_{t \to 0^+} \int_t^1 1/x \, dx$$

$$= \lim \left(\ln 1 - \ln t \right)$$

Side comment

[illegible]

$\lim_{t \to 0^+} (\ln t) = -\infty$

$\lim_{t \to 0^+} (\ln 1 - \ln t)$

$= \lim_{t \to 0^+} (-\ln t)$

$= -(-\infty) = \infty$

we say $\int_0^1 1/x \, dx$ is a divergent improper integral

diverges to ∞.

$\int_1^\infty e^{-x} \, dx$ means

$$\lim_{t \to \infty} \int_1^t e^{-x} \, dx = \lim_{t \to \infty} \left[-e^{-x} \right]_1^t$$

$$= \lim_{t \to \infty} \left(-e^{-t} + e^{-1} \right)$$

convergent improper integral

$$= \lim_{t \to \infty} \left(-\underbrace{\frac{1}{e^t}}_{\to 0} + \frac{1}{e} \right)$$

$$= \frac{1}{e}.$$

In a nutshell - if limits of integration are either

- $\pm\infty$ (type I)
- not in domain of function (type II)

then we

- call it an improper integral
- evaluate by replacing bad bounds w/ a variable, taking a limit.

Convergent means - answer is a number

divergent means - answer is not a number

$$\int_0^\infty \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx = \lim_{t\to\infty}\Big[-\cos x\Big]_0^t$$

$$\lim_{t\to\infty}\left(-\cos t+1\right)$$

d.n.e.

divergent integral.

$$\int_{-1}^{1} 1/x^2\,dx \qquad \text{Improper!}$$

$$= \int_{-1}^{0} 1/x^2\,dx + \int_{0}^{1} 1/x^2\,dx$$

do each part separately.

$$\int_{0}^{1} 1/x^2\,dx = \lim_{t \to 0^+} \int_{t}^{1} 1/x^2\,dx$$

+ to stay in interval from 0 to 1

$$= \lim_{t \to 0^+} \left[-1/x \right]_{t}^{1}$$

$$= \lim_{t \to 0^+} \left[-1/1 + 1/t \right]$$

1/small = big

$$-1 + \text{"}\infty\text{"}$$

= diverges to ∞.

other part $\int_{-1}^{0} 1/x^2 dx = \lim_{t\to 0^-} \int_{-1}^{t} 1/x^2 dx \cdots\cdots = \infty$

in total both pieces diverge to ∞. $\Rightarrow$ in total diverges to ∞.

$$\int_{-1}^{1} x^{-2} dx = \int_{-1}^{0} x^{-2} dx + \int_{0}^{1} x^{-2} dx$$

big + big
"
big

$$\int_{1}^{\infty} 1/x\, dx = \lim_{t\to\infty} \int_{1}^{t} 1/x\, dx = \lim_{t\to\infty} \left[\ln|x|\right]_1^t$$

Improper!

$$= \lim_{t\to\infty} \ln t - \underbrace{\ln 1}_{0}$$

$$= \lim_{t\to\infty} \ln t = \infty$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}$$

(Riemann-Zeta Function)