Lecture 16: trig sub and other things

Tuesday, September 30, 2014 12:30 PM

Practice

1. $\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{\quad}{u}$

...

$u^2 = x^2+1$

$2u\,du = 2x\,dx$

$\sin^2 u + \cos^2 u = 1$

better? $x = \tan u$

$1+\tan^2 u = \sec^2 u$

$dx = \sec^2 u\,du$

$\sec u = \sqrt{1+\tan^2 u}$

$$\int \frac{1}{\sqrt{\tan^2 u+1}} \sec^2 u\,du = \int \frac{1}{\sqrt{\sec^2 u}} \sec^2 u\,du$$

$$\int \frac{1}{\sec u} \sec^2 u\,du = \int \sec u\,du$$

$$= \int \sec u \frac{\sec u + \tan u}{\sec u + \tan u}\,du = \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u}\,du$$

$\int \dfrac{\quad}{\sec u + \tan u} \quad$

$v = \sec u + \tan u$

$dv = (\text{top})\, du$

$$= \int \frac{1}{v}\, dv = \ln|v| + C = \ln|\sec u + \underbrace{\tan u}_{x}| + C$$

$x = \tan u$

$\sec u = ?$

$1 + \tan^2 u = \sec^2 u$

$\sec u = \sqrt{1 + \tan^2 u}$

$= \sqrt{1 + x^2}$

$$\boxed{\ln\left|\sqrt{1+x^2} + x\right| + C}$$

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2. $$\int \frac{1}{\sqrt{25x^2 - 4}}\, dx$$

$$= \int \frac{1}{\sqrt{\frac{4}{4}(25x^2 - 4)}}\, dx = \int \frac{1}{\sqrt{4\left(\frac{25}{4}x^2 - 1\right)}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{25/4\, x^2 - 1}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 x^2 - 1}}\, dx = \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{5}{2}x\right)^2 - 1}}\, dx$$

choice 1: $u = \frac{5}{2}x$

choice 2: $\frac{5}{2}x = \sec u$

choice 3 don't do any of this:

$$\int \frac{1}{\sqrt{25x^2-4}}$$

$$x = \frac{2}{5}\sec u$$

$$\frac{1}{\sqrt{25\left(\frac{2}{5}\sec u\right)^2 - 4}} = \frac{1}{\sqrt{\frac{4\cdot 25}{25}\sec^2 u - 4}}$$

$\sqrt{x^2-1}$

$x = \sec u$

$\sec^2 u - 1 = \tan^2 u$!

$\sin^2 u + \cos^2 u = 1$

$1 - \sin^2 u = \cos^2 u$

$1 + \tan^2 u = \sec^2 u$

$\sqrt{1-x^2}$

$x = \sin u$ or $\cos u$

$\sqrt{1+x^2}$

$x = \tan u$

$x = \sin u$ or $\cos u$

Practice
what to substitute?

1. $\sqrt{x^2-9}$
2. $\sqrt{3x^2-1}$
3. $\sqrt{1-9x^2}$
4. $\sqrt{16+25x^2}$

$\sqrt{4-x^2}$

$x = 2\sin u$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2 u}$
$= \sqrt{4(1-\sin^2 u)}$
$= \sqrt{4\cos^2 u}$
$= 2\cos u$

1. $x = 3\sec u$
2. $x = \frac{1}{\sqrt{3}}\sec u$
3. $x = \frac{1}{3}\cos u$ or $\frac{1}{3}\sin u$
4. $x = \frac{4}{5}\tan u$

Practice

1. $\int \frac{x^2}{1+x^2}\,dx$
2. $\int \frac{1}{\sqrt{9x^2-2}}\,dx$
3. $\int \frac{e^{2x}}{\sqrt{e^{2x}+1}}\,dx$

1. $x = \tan u$

$dx = \sec^2 u \, du$

$$\int \frac{\tan^2 u}{1+\tan^2 u} \sec^2 u \, du$$

$$= \int \tan^2 u \, du$$

$$= \int (\sec^2 u - 1) \, du = \int \sec^2 u \, du - \int du$$

$$= \tan u - u + C$$

$$= x - \arctan x + C$$

2. $$\int \frac{1}{\sqrt{9x^2-2}} \, dx = \frac{\sqrt{2}}{3} \int \frac{\sec u \tan u}{\sqrt{2(\sec^2 u - 1)}} \, du$$

$x = \frac{\sqrt{2}}{3} \sec u \qquad dx = \frac{\sqrt{2}}{3} \sec u \tan u \, du$

$$\frac{\sqrt{2}}{3} \int \frac{1}{\sqrt{2}} \frac{\sec u \tan u}{\sqrt{\tan^2 u}} \, du = \frac{1}{3} \int \sec u \, du$$

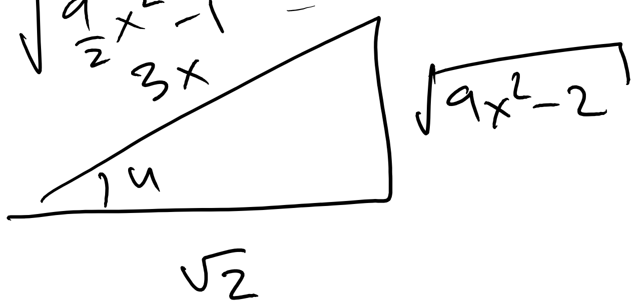
$$= \frac{1}{3} \ln|\sec u + \tan u| + C$$

$$x = \frac{\sqrt{2}}{3} \sec u \rightsquigarrow \sec u = \frac{3}{\sqrt{2}} x$$

$$= \frac{1}{3} \ln\left| \frac{3}{\sqrt{2}} x + \sqrt{\frac{9}{2} x^2 - 1} \right| + C$$

$$\sec u = \frac{3}{\sqrt{2}} x = \frac{3x}{\sqrt{2}}$$

$$\tan u = \frac{\sqrt{9x^2 - 2}}{\sqrt{2}} = \sqrt{\frac{9}{2} x^2 - 1}$$



<u>Steps</u>: ∘ pick substitution (standard patterns from trig identities)

∘ do the trig integral

∘ substitute back for x (Δ's or trig id's)

$$\int \frac{e^{2x}}{\sqrt{e^{2x}+1}}\,dx = \int \frac{e^x e^x}{\sqrt{(e^x)^2+1}}\,dx$$

$$e^{2x} = (e^x)^2$$

$$e^x = \tan u$$

$$e^x\,dx = \sec^2 u\,du$$

$$\int \frac{\tan u \sec^2 u\,du}{\sqrt{\tan^2 u+1}} = \int \frac{\tan u \sec^2 u\,du}{\sqrt{\sec^2 u}}$$

$$= \int \frac{\tan u \sec^2 u}{\sec u}\,du = \int \tan u \sec u\,du$$

$$= \sec u + C$$

$$1+\tan^2 u = \sec^2 u \qquad \sec u = \sqrt{1+\tan^2 u}$$

$$= \sqrt{1+e^{2x}}$$

$$= \sqrt{1+e^{2x}} + C$$

$$\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du$$

$$\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{1}{2}\,2\,u^{1/2} + C = \sqrt{u} + C = \sqrt{1+e^{2x}} + C$$

$$u = 1+e^{2x} \qquad du = 2e^{2x}\,dx$$

<u>Flavor of partial fractions</u>

$$\int \frac{5x-3}{x^2-2x-3}\,dx \overset{\text{algebra}}{\rightsquigarrow} \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$u = x+1$ $\qquad$ $u = x-3$

$2\ln|x+1|$

$3\ln|x-3|$

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

$$\frac{5x-3}{(x+1)(x-3)} = \text{I hope} = \frac{A}{x+1} + \frac{B}{x-3}$$

cross-multiply & solve for A & B.