Lecture 13: the anti-product rule

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Product Rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Backwards: (anti-derivatives of both sides)

$$f(x)g(x) = \int f'(x)g(x)\,dx + \underbrace{\int f(x)g'(x)\,dx}$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Notation: $u = f(x)$ $v = g(x)$

$du = f'(x)\,dx$ $dv = g'(x)\,dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by parts

have to know how to take $u \rightsquigarrow du$ (derivative)

$du \rightsquigarrow v$ (anti-der)

ex: $\int \underset{u}{x}\,\underset{dv}{e^x dx} = \underset{u}{x}\,\underset{v}{e^x} - \int \underset{v}{e^x}\,\underset{du}{dx} = xe^x - e^x + C.$

want to take derivative ↗ ↖ want anti-dv.

$u = x$ $\quad dv = e^x\,dx$

$du = dx$ $\quad v = e^x$ (choose any)

$\int u\,dv = uv - \int v\,du$

ex: $\int x\sin x\,dx = -x\cos x + \int \cos x\,dx$

$u = x$, $dv = \sin x\,dx$ $\rightarrow$ $du = dx$, $v = -\cos x$

$= -x\cos x + \sin x + C$

ex: $\int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$

$u = x^2$, $dv = \sin x\,dx$ $\rightarrow$ $du = 2x\,dx$, $v = -\cos x$

$\rightsquigarrow$ $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

$u = x$, $dv = \cos x\,dx$ $\rightarrow$ $du = dx$, $v = \sin x$

$dv = \cos x\, dx \Big\} \to \quad v = \sin x$

$$= -x^2 \cos x + 2\left(x \sin x + \cos x \right) + C$$

Practice

$\int u\,dv = uv - \int v\,du$

1. $\int x \sin 2x\, dx$

2. $\int x^2 e^{-x}\, dx$

no fear!
keep going!
never give up!

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1. $\int x \sin 2x\, dx = -\frac{1}{2} x \cos 2x + \frac{1}{2}\int \cos 2x\, dx$

$u = x$, $du = dx$

$dv = \sin 2x\, dx$, $v = -\frac{1}{2}\cos 2x$

$$= -\frac{1}{2} x \cos 2x + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + C$$

2. $\int x^4 e^{-x}\, dx = -x^4 e^{-x} + 4\int x^3 e^{-x}\, dx$

$$2. \int x^4 e^{-x}\,dx = -x^4 e^{-x} + 4\int$$

$$\left.\begin{aligned} u &= x^4 \\ dv &= e^{-x}\,dx \end{aligned}\right\} \to \begin{aligned} du &= 4x^3\,dx \\ v &= -e^{-x} \end{aligned}$$

$$\int x^3 e^{-x}\,dx = -x^3 e^{-x} + 3\int x^2 e^{-x}\,dx$$

$$\left.\begin{aligned} u &= x^3 \\ dv &= e^{-x}\,dx \end{aligned}\right\} \to \begin{aligned} du &= 3x^2\,dx \\ v &= -e^{-x} \end{aligned}$$

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\int x e^{-x}\,dx$$

$$\left.\begin{aligned} u &= x^2 \\ dv &= e^{-x}\,dx \end{aligned}\right\} \to \begin{aligned} du &= 2x\,dx \\ v &= -e^{-x} \end{aligned}$$

$$\int x e^{-x}\,dx = -x e^{-x} + \int e^{-x}\,dx = -x e^{-x} - e^{-x}$$

$$\left.\begin{aligned} u &= x \\ dv &= e^{-x}\,dx \end{aligned}\right\} \to \begin{aligned} du &= dx \\ v &= -e^{-x} \end{aligned}$$

$$\int x^4 e^{-x}\,dx = -x^4 e^{-x} + 4\left(-x^3 e^{-x} + 3\left(-x^2 e^{-x} + 2\left(\quad\right)\right)\right)$$

$$-xe^{-x} - e^{-x}$$

$$-x^4 e^{-x} - 4x^3 e^{-x} - 4\cdot 3\, x^2 e^{-x} - 4\cdot 3\cdot 2\, x e^{-x} - 4\cdot 3\cdot 2\cdot 1$$

[illegible] e^{-x}

$$\int x^n e^{-x}\,dx = -e^{-x}x^n + n\int x^{n-1}e^{-x}\,dx$$

$u = x^n$, $dv = e^{-x}\,dx$ $\Big\} \to$ $du = nx^{n-1}$, $v = -e^{-x}$

$$\int x^8 e^{-x}\,dx = -e^{-x}x^8 + 8\cdot\left(-e^{-x}x^7 + 7(-e^{-x} \ldots\right)$$

$$\int \sin^2 x\,dx$$

$u = \sin x$

$du = \cos x\,dx$ . . . – :(

$u = \sin^2 x$

$du = 2\sin x \cos x\,dx$:(

$$\int \sin^2 x\,dx = -\sin x\cos x + \int \cos^2 x\,dx$$

$u = \sin x$, $dv = \sin x\,dx$ $\Big\}$ $du = \cos x\,dx$, $v = -\cos x$

$$= -\sin x\cos x + \int (1-\sin^2 x)\,dx$$

$$\int \sin^2 x\,dx = -\sin x\cos x + \underbrace{\int 1\,dx} - \int \sin^2 x\,dx$$

$$\int \sin^2 x\,dx = -\sin x\cos x + \underbrace{\int dx}_{x} \ldots$$

$$2\int \sin^2 x\,dx = -\sin x\cos x + x + C$$

$$\int \sin^2 x\,dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C.$$

$$\int \sin x\, e^x\,dx = \sin x\, e^x - \int e^x \cos x\,dx$$

$u = \sin x$, $dv = e^x\,dx$ $\rightarrow$ $du = \cos x\,dx$, $v = e^x$

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$$

$u = \cos x$, $dv = e^x\,dx$ $\rightarrow$ $du = -\sin x\,dx$, $v = e^x$

$$\int e^x \sin x\,dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right)$$

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

$$\int e^x \sin x\,dx = \underline{e^x \sin x - e^x \cos x} + C.$$

$$\int v\,dw = vw - \int w\,dv$$

$$\int \ln x\,dx = \int u e^u\,du = ue^u - \int e^u\,du = ue^u - e^u + C$$

$u = \ln x$ $\quad$ $v = u$ $\quad$ $dv = du$

$e^u = x$ $\quad$ $dw = e^u\,du$ $\quad$ $w = e^u$

$e^u\,du = dx$

$$= (\ln x)x - x + C$$

$$\boxed{\int \ln x\,dx = x\ln x - x + C}$$

$$\int \ln x\,dx = x \cdot \ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int dx$$

$u = \ln x$ $\Big\} \rightarrow$ $du = \frac{1}{x}dx$ $\quad = x\ln x - x + C$

$dv = dx$ $\quad$ $v = x$

$$\int v\,dw = vw - \int w\,dv$$

$$\int \arcsin x\,dx = \int u\cos u\,du$$

$v = u$ $\quad$ $dv = du$

$u = \arcsin x$

$\sin u = x$

$\cos u \, du = dx$

$v = u \qquad dv = du$

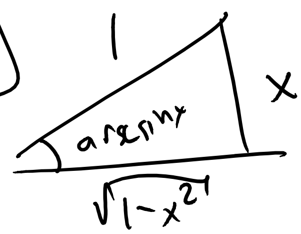
$dw = \cos u \, du \qquad w = \sin u$

$$= u \sin u - \int \sin u \, du$$

$$= u \sin u + \cos u + C$$

$$= (\arcsin x) \underbrace{\sin(\arcsin x)}_{x} + \cos(\arcsin x) + C$$

cosine of $\angle$ whose sine is x



$= \sqrt{1-x^2}$

$$= \boxed{x \arcsin x + \sqrt{1-x^2} + C}$$