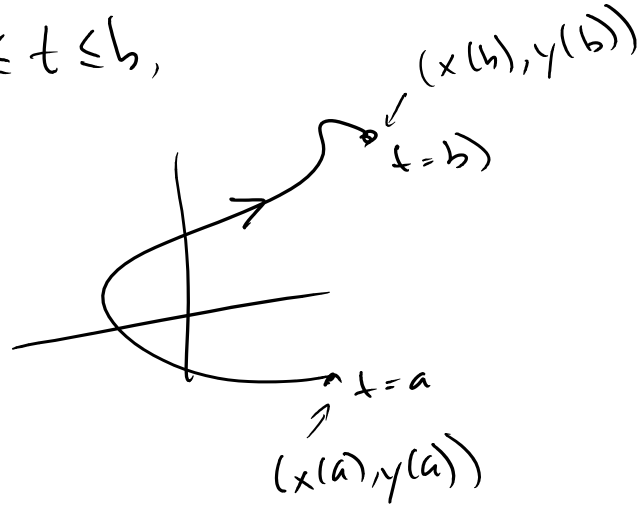
Lecture 10: surfaces of revolution

Tuesday, September 9, 2014 12:34 PM

Recall: If we are given parametric curve

$x(t), y(t)$ $\quad a \le t \le b$,



we can compute

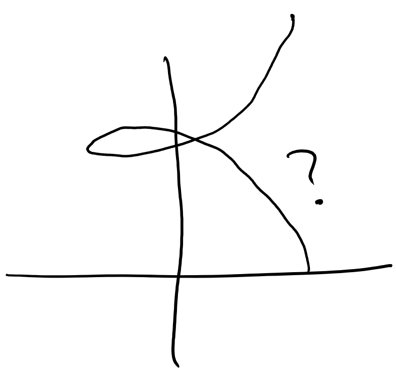
arclength given by

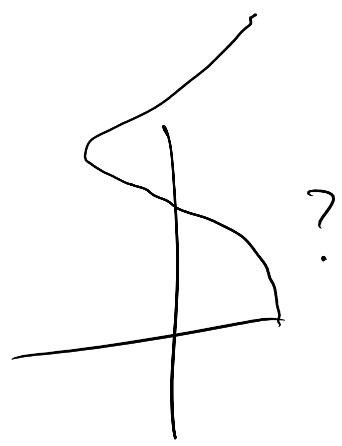
$$\int_{t=a}^{t=b} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Common strategy for evaluating this - get rid of the square root!

usually by making inside a square & cancelling!

ex: $x(t) = \cos t \quad y(t) = \sin t + t \quad 0 \le t \le 2\pi$





$x'(t) = -\sin t \qquad y'(t) = \cos t + 1$

$x'(t)^2 = \sin^2 t \qquad y'(t)^2 = \cos^2 t + 2\cos t + 1$

$$\int_0^{2\pi} \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 2\cos t + 1}\, dt$$

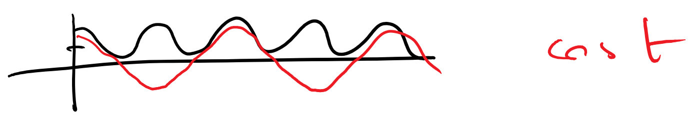
$$= \int_0^{2\pi} \sqrt{2\cos t + 2}\, dt \;\cancel{=}\; \cancel{\int_0^{2\pi} 2\cos\left(\tfrac{1}{2}t\right) dt}$$

$$= \int_0^{2\pi} \sqrt{4\cos^2\left(\tfrac{t}{2}\right)}\, dt$$

$$\cos\left(\tfrac{1}{2}t\right) = \sqrt{\frac{\cos t + 1}{2}}$$

$$2\cos\left(\tfrac{1}{2}t\right) = 2\sqrt{\frac{\cos t + 1}{2}} = \sqrt{4\,\frac{\cos t + 1}{2}} = \sqrt{2\cos t + 2}$$

$$\cos^2 t = \frac{\cos 2t}{2} + \frac{1}{2}$$



$\sin^2 t =$

$$\cos^2 t = \frac{\cos 2t + 1}{2}$$

$$\cos^2 \frac{t}{2} = \frac{\cos t + 1}{2}$$

$$\cos \frac{t}{2} = \sqrt{\frac{\cos t + 1}{}}$$

$$\cos\frac{t}{2} = \sqrt{\frac{\cos t + 1}{2}}$$

come back in a bit

$$y = \ln x - \frac{x^2}{8} \qquad 1 \le x \le 2$$

$$\boxed{x = t}$$

$$x(t) = t \qquad 1 \le t \le 2$$

$$y(t) = \ln t - \frac{t^2}{8} \qquad \int_{t=1}^{t=2} \sqrt{x'(t) + y'(t)}\, dt$$

$$\int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \leftarrow dx = dt$$

$$\frac{dx}{dt} = \frac{dx}{dx} = 1 \qquad \frac{dy}{dt} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{x}{4} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}$$

$$\int_1^2 \sqrt{1 + \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}}\, dx = \int_1^2 \sqrt{\frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16}}$$

$$\int^2 \sqrt{\frac{1}{x^2}\left(1 + \frac{x^2}{2} + \frac{x^4}{16}\right)} = \int^2 \frac{1}{x}\sqrt{\frac{1}{16}\left(16 + 8x^2 + x^4\right)}$$

$$= \int_1^2 \sqrt{\frac{1}{x^2}\left(1 + \frac{x^2}{2} + \frac{x^4}{16}\right)} = \int_1 \frac{1}{x}\sqrt{\frac{1}{16}\left(16 + 8x^2 + x^4\right)}$$

because $x > 0$!

$$= \int_1^2 \frac{1}{4x}\sqrt{(x^2+4)^2}$$

$$= \int_1^2 \frac{1}{4x}(x^2+4)\,dx = \int_1^2 \left(\frac{1}{4}x + \frac{1}{x}\right)dx$$

because $x^2+4 > 0$!

$$= \left[\frac{1}{8}x^2 + \ln|x|\right]_1^2$$

$$\left(\frac{1}{8}(2)^2 + \ln 2\right) - \left(\frac{1}{8}(1)^2 + \ln 1\right)$$

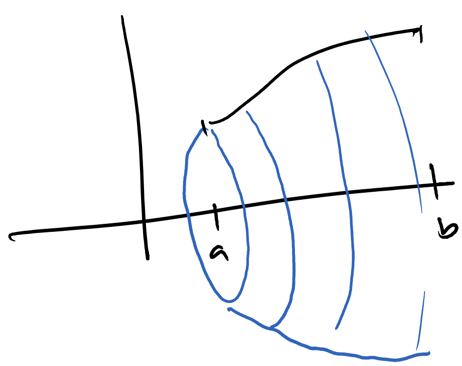
$$\frac{1}{2} + \ln 2 - \frac{1}{8} - 0$$

$$= \frac{3}{8} + \ln 2$$

Surface areas of Revolution

$y = f(x)$

Setup:

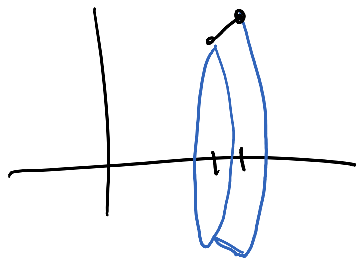


revolve about x-axis

want to measure surface area

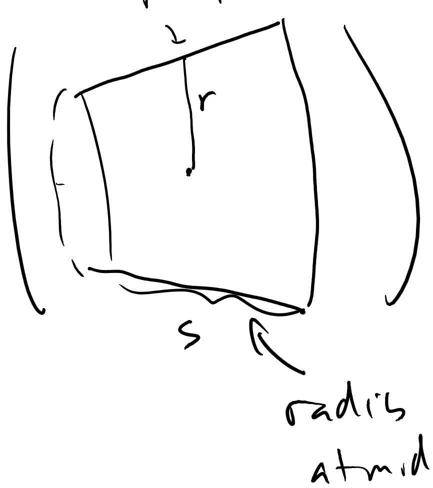
(not volume)

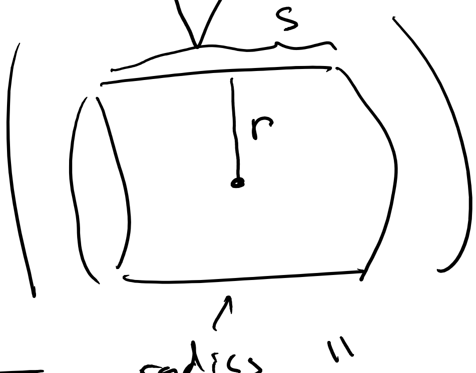
<u>basic idea:</u> break up surface areas into angled rings



idea: add these up!
"frustum of cone"

<u>Fact:</u>

SA 

= SA 

$= 2\pi r s$

for us $s \approx$ piece of arclength. $\approx \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$r \approx$ ht of function $= f(x) = y$

$$SA = \int_{x=a}^{x=b} \text{frustum s.a.}(x)\,dx = \int_a^b 2\pi r s\,dx$$

$$\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

$x=a$

$$\boxed{SA = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx}$$

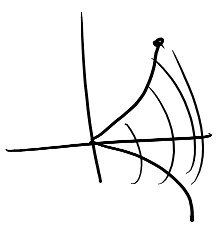
$$\int_{x=a}^{x=b} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

about y-axis

$$SA = \int_{y=a}^{y=b} 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Typically hard: 1. $y=x^2$ about x

$0 \le x \le 1$

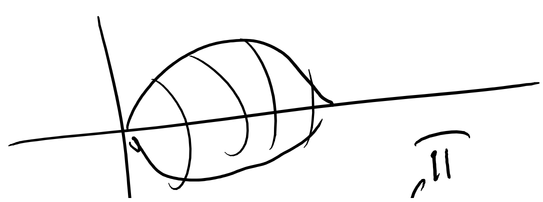


$$\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

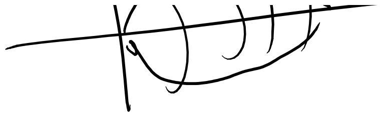
$$= \int_0^1 2\pi x^2\sqrt{1+(2x)^2}\,dx$$

$$= \int_0^1 2\pi x^2\sqrt{1+4x^2}\,dx$$

$y=\sin x$ $0 \le x \le \pi$ about x-axis



$$\int^{\pi} \quad \sqrt{1+ \quad)^2}\,dx$$



$$\int_0^{\pi} 2\pi y \sqrt{1+y'^2}\, dx$$

$$y' = \cos x$$

$$2\pi \int_0^{\pi} \sin x \sqrt{1+\cos^2 x}\, dx$$

$$1+\tan^2 = \sec^2$$

$$\cos x = \tan u$$

$$\frac{d}{dx}\cos x = \frac{d}{dx}\tan u$$

$$-\sin x = \sec^2 u \frac{du}{dx}$$

$$-\sin x\, dx = \sec^2 u\, du$$

$$-2\pi \int_?^? \sec^2 u \sqrt{1+\tan^2 u}\, du$$

$$= -2\pi \int_?^? \sec^3 u\, du$$

$$y = 2\sqrt{x} = 2x^{\frac{1}{2}} \quad 1 \le x \le 2$$

$$\int^2 2\pi y \sqrt{1+y'^2}\, dx = \int_1^2 2\pi\, 2\sqrt{x} \sqrt{1+\frac{1}{x}}\, dx$$

$$y' = x^{-1/2} = \frac{1}{\sqrt{x}} \qquad y'^2 = \frac{1}{x}$$

$$4\pi \int_1^2 \sqrt{x} \sqrt{\frac{x}{x} + \frac{1}{x}}\, dx = 4\pi \int_1^2 \sqrt{x} \sqrt{\frac{x+1}{x}}\, dx$$

$$= 4\pi \int_1^2 \cancel{\sqrt{x}}\, \frac{\sqrt{x+1}}{\cancel{\sqrt{x}}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$x = 1 \rightsquigarrow u = 2$

$u = x + 1$

$du = dx$

$x = 2 \rightsquigarrow u = 3$

$$4\pi \int_2^3 u^{1/2}\, du = 4\pi \left[\frac{2}{3} u^{3/2}\right]_2^3$$

$$= 4\pi \cdot \frac{2}{3} \left[3^{3/2} - 2^{3/2}\right]$$

$$\frac{8\pi}{3} \left(\sqrt{27} - \sqrt{8}\right)$$